CHAPTER

9

PARALLEL LINES

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"If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, then the two straight lines, if produced indefinitely, meet on that side on which the angles are less than two right angles."

This statement, Euclid's fifth postulate, is called **Euclid's parallel postulate**. Throughout history this postulate has been questioned by mathematicians because many felt it was too complex to be a postulate.

Throughout the history of mathematics, attempts were made to prove this postulate or state a related postulate that would make it possible to prove Euclid's parallel postulate. Other postulates have been proposed that appear to be simpler and which could provide the basis for a proof of the parallel postulate.

The form of the parallel postulate most commonly used in the study of elementary geometry today was proposed by John Playfair (1748–1819). **Playfair's postulate** states:

- Through a point not on a given line there can be drawn one and only one line parallel to the given line.

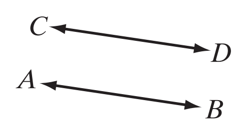
9-1 PROVING LINES PARALLEL

You have already studied many situations involving intersecting lines that lie in the same plane. When all the points or lines in a set lie in a plane, we say that these points or these lines are **coplanar**. Let us now consider situations involving coplanar lines that do not intersect in one point.

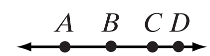
DEFINITION

Parallel lines are coplanar lines that have no points in common, or have all points in common and, therefore, coincide.

The word "lines" in the definition means straight lines of unlimited extent. We say that segments and rays are parallel if the lines that contain them are parallel.



We indicate that $\overleftrightarrow{AB}$ is parallel to $\overleftrightarrow{CD}$ by writing $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$. The parallel lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ extended indefinitely never intersect and have no points in common.



The parallel lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ may have all points in common, that is, be two different names for the same line. A line is parallel to itself. Thus, $\overleftrightarrow{AB} \parallel \overleftrightarrow{AB}$, $\overleftrightarrow{CD} \parallel \overleftrightarrow{CD}$ and $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$.

In Chapter 4, we stated the following postulate:

► **Two distinct lines cannot intersect in more than one point.**

This postulate, together with the definition of parallel lines, requires that one of three possibilities exist for any two coplanar lines, $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$:

1. $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ have no points in common.
$\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel.

2. $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ have only one point in common.
$\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect.

3. $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ have all points in common.
$\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are the same line.

These three possibilities can also be stated in the following postulate:

Postulate 9.1

Two distinct coplanar lines are either parallel or intersecting.

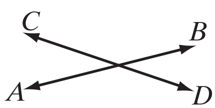
EXAMPLE 1

If line l is not parallel to line p, what statements can you make about these two lines?

Solution Since l is not parallel to p, l and p cannot be the same line, and they have exactly one point in common. *Answer*

Parallel Lines and Transversals

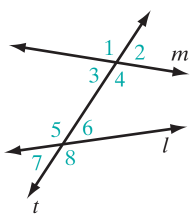
$\overleftrightarrow{AB}$ intersects $\overleftrightarrow{CD}$



When two lines intersect, four angles are formed that have the same vertex and no common interior points. In this set of four angles, there are two pair of congruent vertical angles and four pair of supplementary adjacent angles. When two lines are intersected by a third line, two such sets of four angles are formed.

DEFINITION

A **transversal** is a line that intersects two other coplanar lines in two different points.



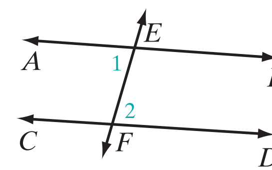
Two lines, l and m, are cut by a transversal, t. Two sets of angles are formed, each containing four angles. Each of these angles has one ray that is a subset of l or of m and one ray that is a subset of t. In earlier courses, we learned names to identify these sets of angles.

- The angles that have a part of a ray between l and m are **interior angles**.
  Angles 3, 4, 5, 6 are interior angles.
- The angles that do not have a part of a ray between l and m are **exterior angles**.
  Angles 1, 2, 7, 8 are exterior angles.
- **Alternate interior angles** are on opposite sides of the transversal and do not have a common vertex.
  Angles 3 and 6 are alternate interior angles, and angles 4 and 5 are alternate interior angles.
- **Alternate exterior angles** are on opposite sides of the transversal and do not have a common vertex.
  Angles 1 and 8 are alternate exterior angles, and angles 2 and 7 are alternate exterior angles.
- **Interior angles on the same side of the transversal** do not have a common vertex.
  Angles 3 and 5 are interior angles on the same side of the transversal, and angles 4 and 6 are interior angles on the same side of the transversal.
- **Corresponding angles** are one exterior and one interior angle that are on the same side of the transversal and do not have a common vertex.
  Angles 1 and 5, angles 2 and 6, angles 3 and 7, and angles 4 and 8 are pairs of corresponding angles.

In the diagram shown on page 330, the two lines cut by the transversal are not parallel lines. However, when two lines are parallel, many statements may be postulated and proved about these angles.

Theorem 9.1a

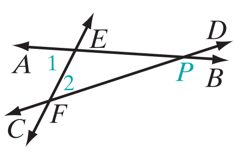
> If two coplanar lines are cut by a transversal so that the alternate interior angles formed are congruent, then the two lines are parallel.



Given $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are cut by transversal $\overleftrightarrow{EF}$ at points E and F, respectively; $\angle 1 \cong \angle 2$.

Prove $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$

Proof To prove this theorem, we will use an indirect proof.



Statements	***Reasons***
1. $\overleftrightarrow{AB}$ is not parallel to $\overleftrightarrow{CD}$.	**1.** Assumption.
2. $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are cut by transversal $\overleftrightarrow{EF}$ at points E and F, respectively.	**2.** Given.
3. $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at some point P, forming $\triangle EFP$.	**3.** Two distinct coplanar lines are either parallel or intersecting.
4. $m\angle 1 > m\angle 2$	**4.** The measure of an exterior angle of a triangle is greater than the measure of either nonadjacent interior angle.
5. But $\angle 1 \cong \angle 2$.	**5.** Given.
6. $m\angle 1 = m\angle 2$	**6.** Congruent angles are equal in measure.
7. $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$	**7.** Contradiction in steps 4 and 6.

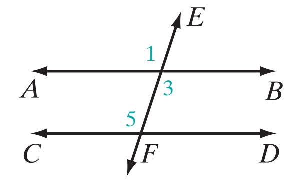
Now that we have proved Theorem 9.1, we can use it in other theorems that also prove that two lines are parallel.

Theorem 9.2a

> If two coplanar lines are cut by a transversal so that the corresponding angles are congruent, then the two lines are parallel.

Given $\overleftrightarrow{EF}$ intersects $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$; $\angle 1 \cong \angle 5$.

Prove $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$



Proof

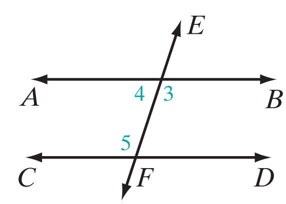
Statements	Reasons
1. $\overleftrightarrow{EF}$ intersects $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$; $\angle 1 \cong \angle 5$	**1.** Given.
2. $\angle 1 \cong \angle 3$	**2.** Vertical angles are congruent.
3. $\angle 3 \cong \angle 5$	**3.** Transitive property of congruence.
4. $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$	**4.** If two coplanar lines are cut by a transversal so that the alternate interior angles formed are congruent, then the two lines are parallel.

Theorem 9.3a

> If two coplanar lines are cut by a transversal so that the interior angles on the same side of the transversal are supplementary, then the lines are parallel.

Given $\overleftrightarrow{EF}$ intersects $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$, and $\angle 5$ is the supplement of $\angle 4$.

Prove $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$



Proof Angle 4 and angle 3 are supplementary since they form a linear pair. If two angles are supplements of the same angle, then they are congruent. Therefore, $\angle 3 \cong \angle 5$. Angles 3 and 5 are a pair of congruent alternate interior angles. If two coplanar lines are cut by a transversal so that the alternate interior angles formed are congruent, then the lines are parallel. Therefore, $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$.

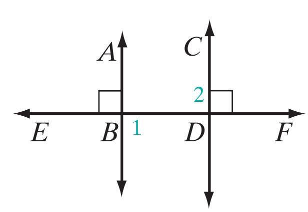
Theorem 9.4

> If two coplanar lines are each perpendicular to the same line, then they are parallel.

Given $\overleftrightarrow{AB} \perp \overleftrightarrow{EF}$ and $\overleftrightarrow{CD} \perp \overleftrightarrow{EF}$.

Prove $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$

Strategy Show that a pair of alternate interior angles are congruent.



The proof of Theorem 9.4 is left to the student. (See exercise 10.)

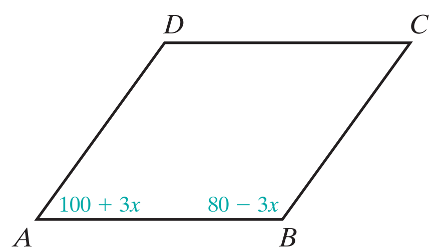
Methods of Proving Lines Parallel

To prove that two coplanar lines that are cut by a transversal are parallel, prove that any one of the following statements is true:

1. A pair of alternate interior angles are congruent.
2. A pair of corresponding angles are congruent.
3. A pair of interior angles on the same side of the transversal are supplementary.
4. Both lines are perpendicular to the same line.

EXAMPLE 2

If $m\angle A = 100 + 3x$ and $m\angle B = 80 - 3x$, explain why $\overline{AD} \parallel \overline{BC}$.



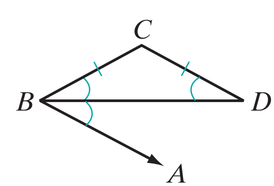
Solution

$$\begin{aligned} m\angle A + m\angle B &= 100 + 3x + 80 - 3x \\ &= 100 + 80 + 3x - 3x \\ &= 180 \end{aligned}$$

Thus, $\angle A$ and $\angle B$ are supplementary. Since $\overline{AD}$ and $\overline{BC}$ are cut by transversal $\overline{AB}$ to form supplementary interior angles on the same side of the transversal, the segments are parallel, namely, $\overline{AD} \parallel \overline{BC}$.

EXAMPLE 3

If $\overline{BD}$ bisects $\angle ABC$, and $\overline{BC} \cong \overline{CD}$, prove $\overline{CD} \parallel \overrightarrow{BA}$.



Proof

(1) Since $\overline{BC} \cong \overline{CD}$, $\angle CBD \cong \angle D$ because the base angles of an isosceles triangle are congruent.

(2) Since $\overline{BD}$ bisects $\angle ABC$, $\angle CBD \cong \angle DBA$ because the bisector of an angle divides the angle into two congruent angles.

(3) Therefore, by the transitive property of congruence, $\angle DBA \cong \angle D$.

(4) Then, $\angle DBA$ and $\angle D$ are congruent alternate interior angles when $\overline{CD}$ and $\overrightarrow{BA}$ are intersected by transversal $\overline{BD}$. Therefore, $\overline{CD} \parallel \overrightarrow{BA}$ because if two coplanar lines are cut by a transversal so that the alternate interior angles formed are congruent, then the two lines are parallel.

Exercises

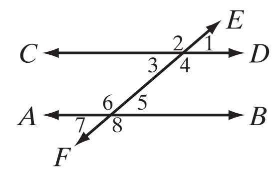
Writing About Mathematics

1. Two lines are cut by a transversal. If $\angle 1$ and $\angle 2$ are vertical angles and $\angle 1$ and $\angle 3$ are alternate interior angles, what type of angles do $\angle 2$ and $\angle 3$ form?

2. Is it true that if two lines that are not parallel are cut by a transversal, then the alternate interior angles are not congruent? Justify your answer.

Developing Skills

In 3–8, the figure shows eight angles formed when $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are cut by transversal $\overleftrightarrow{EF}$. For each of the following, state the theorem or theorems that prove $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$.



3. $m\angle 3 = 70$ and $m\angle 5 = 70$

4. $m\angle 2 = 140$ and $m\angle 6 = 140$

5. $m\angle 3 = 60$ and $m\angle 6 = 120$

6. $m\angle 2 = 150$ and $m\angle 5 = 30$

7. $m\angle 2 = 160$ and $m\angle 8 = 160$

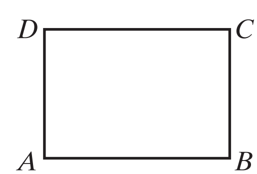
8. $m\angle 4 = 110$ and $m\angle 7 = 70$

Applying Skills

9. Write an *indirect* proof of Theorem 9.2a, "If two coplanar lines are cut by a transversal so that the corresponding angles are congruent, then the two lines are parallel."

10. Prove Theorem 9.4, "If two coplanar lines are each perpendicular to the same line, then they are parallel."

In 11 and 12, $ABCD$ is a quadrilateral.



11. If $m\angle A = 3x$ and $m\angle B = 180 - 3x$. Show that $\overline{AD} \parallel \overline{BC}$.

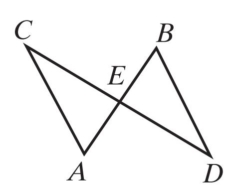
12. If $\overline{DC} \perp \overline{BC}$ and $m\angle ADC = 90$, prove $\overline{AD} \parallel \overline{BC}$.

13. If $\overline{AB}$ and $\overline{CD}$ bisect each other at point E, prove:

a. $\triangle CEA \cong \triangle DEB$

b. $\angle ECA \cong \angle EDB$

c. $\overline{CA} \parallel \overline{DB}$



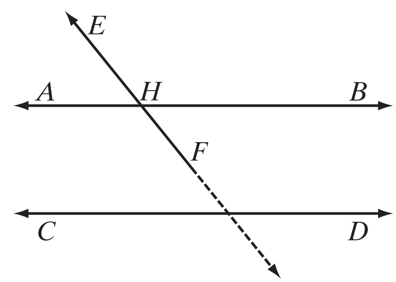
14. Prove that if two coplanar lines are cut by a transversal, forming a pair of alternate exterior angles that are congruent, then the two lines are parallel.

9-2 PROPERTIES OF PARALLEL LINES

In the study of logic, we learned that a conditional and its converse do not always have the same truth value. Once a conditional statement has been proved to be true, it may be possible to prove that its converse is also true. In this section, we will prove converse statements of some of the theorems proved in the previous section. The proof of these converse statements requires the following postulate and theorem:

Postulate 9.2 Through a given point not on a given line, there exists one and only one line parallel to the given line.

Theorem 9.5 If, in a plane, a line intersects one of two parallel lines, it intersects the other.



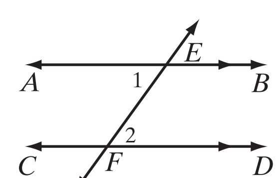
Given $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ and $\overleftrightarrow{EF}$ intersects $\overleftrightarrow{AB}$ at H.

Prove $\overleftrightarrow{EF}$ intersects $\overleftrightarrow{CD}$.

Proof Assume $\overleftrightarrow{EF}$ does not intersect $\overleftrightarrow{CD}$. Then $\overleftrightarrow{EF} \parallel \overleftrightarrow{CD}$. Therefore, through H, a point not on $\overleftrightarrow{CD}$, two lines, $\overleftrightarrow{AB}$ and $\overleftrightarrow{EF}$ are each parallel to $\overleftrightarrow{CD}$. This contradicts the postulate that states that through a given point not on a given line, one and only one line can be drawn parallel to a given line. Since our assumption leads to a contradiction, the assumption must be false and its negation, $\overleftrightarrow{EF}$ intersects $\overleftrightarrow{CD}$ must be true. ∎

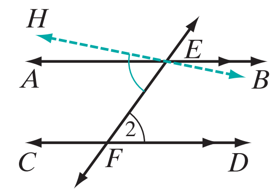
Now we are ready to prove the converse of Theorem 9.1a.

Theorem 9.1b If two parallel lines are cut by a transversal, then the alternate interior angles formed are congruent.



Given $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, transversal $\overleftrightarrow{EF}$ intersects $\overleftrightarrow{AB}$ at E and $\overleftrightarrow{CD}$ at F.

Prove $\angle 1 \cong \angle 2$



Proof We can use an indirect proof. Assume $\angle 1$ is not congruent to $\angle 2$. Construct $\overrightarrow{EH}$ so that $\angle HEF \cong \angle 2$. Since $\angle HEF$ and $\angle 2$ are congruent alternate interior angles, $\overleftrightarrow{HE} \parallel \overleftrightarrow{CD}$. But $\overleftrightarrow{AB}$ is a line through E, and we are given $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$. This contradicts the postulate that states that through a given point not on a given line, there exists one and only one line parallel to the given line. Thus, the assumption is false and $\angle 1 \cong \angle 2$.

Note that Theorem 9.1b is the converse of Theorem 9.1a. We may state the two theorems in biconditional form:

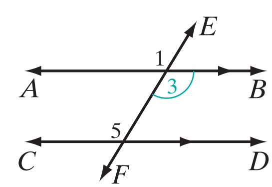
Theorem 9.1 Two coplanar lines cut by a transversal are parallel if and only if the alternate interior angles formed are congruent.

Each of the next two theorems is also a converse of a theorem stated in Section 9-1.

Theorem 9.2b If two parallel lines are cut by a transversal, then the corresponding angles are congruent. (Converse of Theorem 9.2a)

Given $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ and transversal $\overleftrightarrow{EF}$

Prove $\angle 1 \cong \angle 5$



Proof

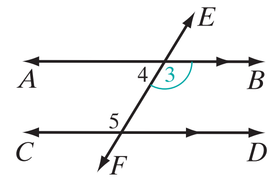
Statements	Reasons
1. $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ and transversal $\overleftrightarrow{EF}$	**1.** Given.
2. $\angle 3 \cong \angle 5$	**2.** If two parallel lines are cut by a transversal, then the alternate interior angles formed are congruent.
3. $\angle 1 \cong \angle 3$	**3.** Vertical angles are congruent.
4. $\angle 1 \cong \angle 5$	**4.** Transitive property of congruence.

Theorem 9.3b If two parallel lines are cut by a transversal, then two interior angles on the same side of the transversal are supplementary. (Converse of Theorem 9.3a)

Given $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ and transversal $\overleftrightarrow{EF}$

Prove $\angle 4$ is the supplement of $\angle 5$.

Strategy Show that $\angle 3 \cong \angle 5$ and that $\angle 4$ is the supplement of $\angle 3$. If two angles are congruent, then their supplements are congruent. Therefore, $\angle 4$ is also the supplement of $\angle 5$.



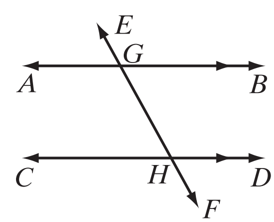
The proof of this theorem is left to the student. (See exercise 18.) Since Theorems 9.2b and 9.3b are converses of Theorems 9.2a and 9.3a, we may state the theorems in biconditional form:

Theorem 9.2 Two coplanar lines cut by a transversal are parallel if and only if corresponding angles are congruent.

Theorem 9.3 Two coplanar lines cut by a transversal are parallel if and only if interior angles on the same side of the transversal are supplementary.

EXAMPLE 1

Transversal $\overleftrightarrow{EF}$ intersects $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ at G and H, respectively. If $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, $m\angle BGH = 3x - 20$, and $m\angle GHC = 2x + 10$:



a. Find the value of x. **b.** Find $m\angle GHC$.

c. Find $m\angle GHD$.

Solution **a.** Since $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ and these lines are cut by transversal $\overleftrightarrow{EF}$, the alternate interior angles are congruent: $m\angle BGH = m\angle GHC$

$$3x - 20 = 2x + 10$$
$$3x - 2x = 10 + 20$$
$$x = 30$$

b. $m\angle GHC = 2x + 10$

$$= 2(30) + 10$$
$$= 70$$

c. Since $\angle GHC$ and $\angle GHD$ form a linear pair and are supplementary,

$$\begin{aligned} m\angle GHD &= 180 - m\angle GHC \\ &= 180 - 70 \\ &= 110 \end{aligned}$$

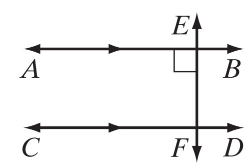
Answers **a.** $x = 70$ **b.** $m\angle GHC = 70$ **c.** $m\angle GHD = 110$

Using Theorem 9.1, we may also prove the following theorems:

Theorem 9.6 If a transversal is perpendicular to one of two parallel lines, it is perpendicular to the other.

Given $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, $\overleftrightarrow{EF} \perp \overleftrightarrow{AB}$

Prove $\overleftrightarrow{EF} \perp \overleftrightarrow{CD}$



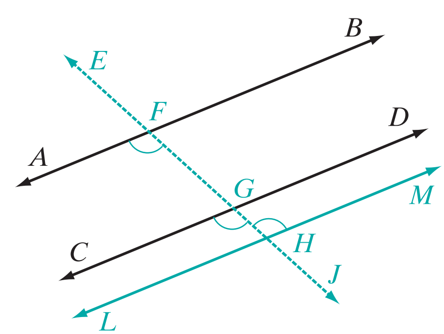
Strategy Show that alternate interior angles are right angles.

The proof of this theorem is left to the student. (See exercise 19.)

Theorem 9.7 If two of three lines in the same plane are each parallel to the third line, then they are parallel to each other.

Given $\overleftrightarrow{AB} \parallel \overleftrightarrow{LM}$ and $\overleftrightarrow{CD} \parallel \overleftrightarrow{LM}$

Prove $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$



Proof Draw transversal $\overleftrightarrow{EJ}$ intersecting $\overleftrightarrow{LM}$ at H. Since $\overleftrightarrow{AB} \parallel \overleftrightarrow{LM}$, this transversal also intersects $\overleftrightarrow{AB}$. Call this point F. Similarly, since $\overleftrightarrow{CD} \parallel \overleftrightarrow{LM}$, this transversal also intersects $\overleftrightarrow{CD}$ at a point G.

Since $\overleftrightarrow{AB} \parallel \overleftrightarrow{LM}$, alternate interior angles formed are congruent. Therefore, $\angle AFG \cong \angle GHM$. Similarly, since $\overleftrightarrow{CD} \parallel \overleftrightarrow{LM}$, $\angle CGH \cong \angle GHM$. By the transitive property of congruence, $\angle AFG \cong \angle CGH$. Angles AFG and CGH are congruent corresponding angles when $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are intersected by transversal

$\overleftrightarrow{EJ}$. Therefore, $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ because if two coplanar lines are cut by a transversal so that the corresponding angles formed are congruent, then the two lines are parallel.

SUMMARY OF PROPERTIES OF PARALLEL LINES

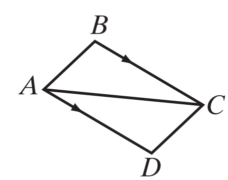
If two lines are parallel:

1. A transversal forms congruent alternate interior angles.
2. A transversal forms congruent corresponding angles.
3. A transversal forms supplementary interior angles on the same side of the transversal.
4. A transversal perpendicular to one line is also perpendicular to the other.
5. A third line in the same plane that is parallel to one of the lines is parallel to the other.

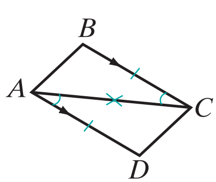
EXAMPLE 2

Given: Quadrilateral $ABCD$, $\overline{BC} \cong \overline{DA}$, and $\overline{BC} \parallel \overline{DA}$

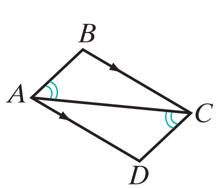
Prove: $\overline{AB} \parallel \overline{CD}$



Proof Use congruent triangles to prove congruent alternate interior angles.



Statements	***Reasons***
1. $\overline{BC} \cong \overline{DA}$	**1.** Given.
2. $\overline{BC} \parallel \overline{DA}$	**2.** Given.
3. $\angle BCA \cong \angle DAC$	**3.** If two parallel lines are cut by a transversal, the alternate interior angles are congruent.
4. $\overline{AC} \cong \overline{AC}$	**4.** Reflexive property of congruence.
5. $\triangle BAC \cong \triangle DCA$	**5.** SAS.
6. $\angle BAC \cong \angle DCA$	**6.** Corresponding parts of congruent triangles are congruent.
7. $\overline{AB} \parallel \overline{CD}$	**7.** If two lines cut by a transversal form congruent alternate interior angles, the lines are parallel.



Note: In the diagram for Example 2, you may have noticed that two parallel lines, $\overleftrightarrow{BC}$ and $\overleftrightarrow{DA}$, each contained a single arrowhead in the same direction. Such pairs of arrowheads are used on diagrams to indicate that two lines are parallel.

Exercises

Writing About Mathematics

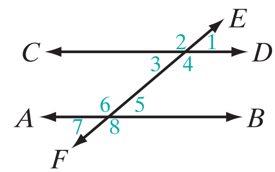
1. a. Is the inverse of Theorem 9.1a always true? Explain why or why not.

b. Is the inverse of Theorem 9.6 always true? Explain why or why not.

2. Two parallel lines are cut by a transversal forming alternate interior angles that are supplementary. What conclusion can you draw about the measures of the angles formed by the parallel lines and the transversal. Justify your answer.

Developing Skills

In 3–12, $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ are cut by transversal $\overleftrightarrow{EF}$ as shown in the diagram. Find:



3. $m\angle 5$ when $m\angle 3 = 80$.

4. $m\angle 2$ when $m\angle 6 = 150$.

5. $m\angle 4$ when $m\angle 5 = 60$.

6. $m\angle 7$ when $m\angle 1 = 75$.

7. $m\angle 8$ when $m\angle 3 = 65$.

8. $m\angle 5$ when $m\angle 2 = 130$.

9. $m\angle 3$ when $m\angle 3 = 3x$ and $m\angle 5 = x + 28$.

10. $m\angle 5$ when $m\angle 3 = x$ and $m\angle 4 = x + 20$.

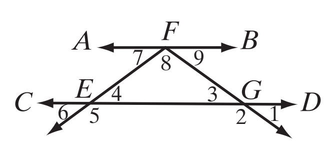
11. $m\angle 7$ when $m\angle 1 = x + 40$ and $m\angle 2 = 5x - 10$.

12. $m\angle 5$ when $m\angle 2 = 7x - 20$ and $m\angle 8 = x + 100$.

13. Two parallel lines are cut by a transversal. For each pair of interior angles on the same side of the transversal, the measure of one angle exceeds the measure of twice the other by 48 degrees. Find the measures of one pair of interior angles.

14. Two parallel lines are cut by a transversal. The measure of one of the angles of a pair of corresponding angles can be represented by 42 less than three times the other. Find the measures of the angles of this pair of corresponding angles.

15. In the diagram, $\overleftrightarrow{AFB} \parallel \overleftrightarrow{CD}$ and $\overleftrightarrow{EF}$ and $\overleftrightarrow{GF}$ intersect $\overleftrightarrow{AB}$ at F.



a. If $m\angle FGD = 110$ and $m\angle FEC = 130$, find the measures of each of the angles numbered 1 through 9.

b. What is the measure of an exterior angle of $\triangle EFG$ at F?

c. Is the measure of an exterior angle at F greater than the measure of either of the nonadjacent interior angles?

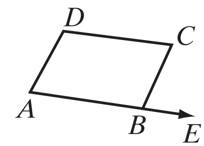
d. What is the sum of the measures of the nonadjacent interior angles of an exterior angle at F?

e. What is the sum of the measures of the nonadjacent interior angles of the exterior angle, $\angle FGD$?

f. What is the sum of the measures of the nonadjacent interior angles of the exterior angle, $\angle FEC$?

g. What is the sum of the measures of the angles of $\triangle EFG$?

16. Two pairs of parallel lines are drawn; $\overleftrightarrow{ABE} \parallel \overleftrightarrow{DC}$ and $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$. If $m\angle CBE = 75$, find the measure of each angle of quadrilateral $ABCD$.



Applying Skills

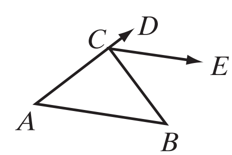
17. Prove Theorem 9.3b, "If two parallel lines are cut by a transversal, then two interior angles on the same side of the transversal are supplementary."

18. Prove Theorem 9.6, "If a transversal is perpendicular to one of two parallel lines, it is perpendicular to the other."

19. Prove that if two parallel lines are cut by a transversal, the alternate exterior angles are congruent.

20. *Given:* $\triangle ABC$, $\overrightarrow{CE}$ bisects exterior $\angle BCD$, and $\overrightarrow{CE} \parallel \overline{AB}$.

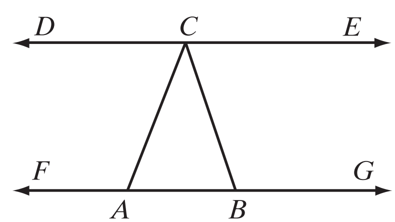
Prove: $\angle A \cong \angle B$.



21. *Given:* $\angle CAB \cong \angle DCA$ and $\angle DCA \cong \angle ECB$

Prove: **a.** $\overline{AB} \parallel \overleftrightarrow{DCE}$.

b. $\angle CAB$ is the supplement of $\angle CBG$.



22. The opposite sides of quadrilateral $PQRS$ are parallel, that is, $\overline{PQ} \parallel \overline{RS}$ and $\overline{QR} \parallel \overline{SP}$. If $\angle P$ is a right angle, prove that $\angle Q$, $\angle R$, and $\angle S$ are right angles.

23. The opposite sides of quadrilateral $KLMN$ are parallel, that is, $\overline{KL} \parallel \overline{MN}$ and $\overline{LM} \parallel \overline{NK}$. If $\angle K$ is an acute angle, prove that $\angle M$ is an acute angle and that $\angle L$ and $\angle N$ are obtuse angles.

9-3 PARALLEL LINES IN THE COORDINATE PLANE

In Chapter 6 we stated postulates about horizontal and vertical lines in the coordinate plane. One of these postulates states that each vertical line is perpendicular to each horizontal line. We can use this postulate to prove the following theorem:

Theorem 9.8

> If two lines are vertical lines, then they are parallel.

Proof: Since each vertical line is perpendicular to each horizontal line, each vertical line is perpendicular to the x-axis, a horizontal line. Theorem 9.6 states that if two coplanar lines are each perpendicular to the same line, then they are parallel. Therefore, all vertical lines are parallel. ☐

A similar theorem can be proved about horizontal lines:

Theorem 9.9

> If two lines are horizontal lines, then they are parallel.

Proof: Since each horizontal line is perpendicular to each vertical line, each horizontal line is perpendicular to the y-axis, a vertical line. Theorem 9.6 states that if two coplanar lines are each perpendicular to the same line, then they are parallel. Therefore, all horizontal lines are parallel. ☐

We know that all horizontal lines have the same slope, 0. We also know that all vertical lines have no slope.

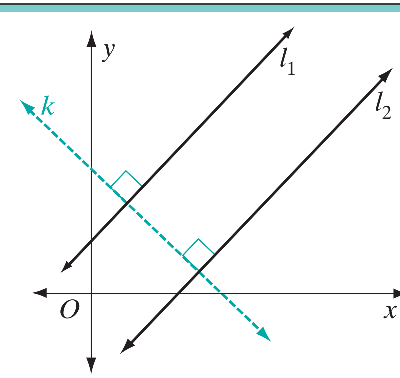
Do parallel lines that are neither horizontal nor vertical have the same slope? When we draw parallel lines in the coordinate plane, it appears that this is true.

Theorem 9.10a

> If two non-vertical lines in the same plane are parallel, then they have the same slope.

Given $l_1 \parallel l_2$

Prove The slope of l_1 is equal to slope of l_2.



Proof In the coordinate plane, let the slope of l_1 be $m \neq 0$. Choose any point on l_1. Through a given point, one and only one line can be drawn perpendicular to a given line. Through that point, draw k, a line perpendicular to l_1.

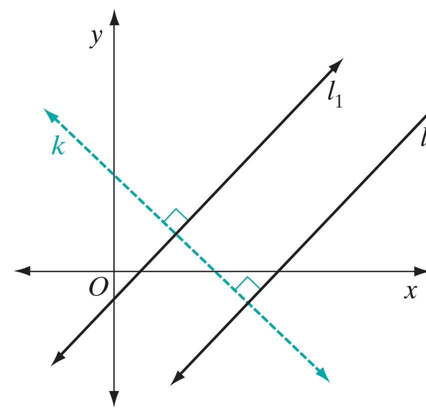
If two lines are perpendicular, the slope of one is the negative reciprocal of the slope of the other. Therefore, the slope of k is $-\frac{1}{m}$. It is given that $l_1 \parallel l_2$. Then, k is perpendicular to l_2 because if a line is perpendicular to one of two parallel lines, then it is perpendicular to the other. The slope of l_2 is the negative reciprocal of the slope of k. The negative reciprocal of $-\frac{1}{m}$ is m. Therefore, the slope of l_1 is equal to the slope of l_2. ☐

Is the converse of this statement true? We will again use the fact that two lines are perpendicular if and only if the slope of one is the negative reciprocal of the slope of the other to prove that it is.

Theorem 9.10b If the slopes of two non-vertical lines in the coordinate plane are equal, then the lines are parallel.

Given Lines l_1 and l_2 with slope m

Prove $l_1 \parallel l_2$



Proof Choose any point on l_1. Through a given point, one and only one line can be drawn perpendicular to a given line. Through that point, draw k, a line perpendicular to l_1. The slope of k is $-\frac{1}{m}$ since two non-vertical lines are perpendicular if and only if the slope of one is the negative reciprocal of the slope of the other. But this means that $l_2 \perp k$ because the slope of l_2 is also the negative reciprocal of the slope of k. Therefore, $l_1 \parallel l_2$ because two lines perpendicular to the same line are parallel. ☐

We can write the statements that we have proved as a biconditional:

Theorem 9.10 Two non-vertical lines in the coordinate plane are parallel if and only if they have the same slope.

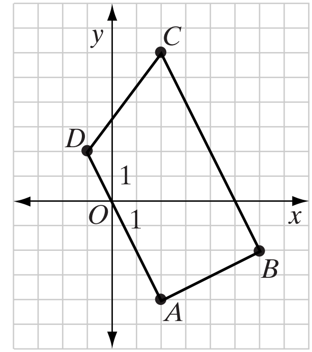
EXAMPLE 1

The vertices of quadrilateral $ABCD$ are $A(2, -4)$, $B(6, -2)$, $C(2, 6)$, and $D(-1, 2)$.

a. Show that two sides of the quadrilateral are parallel.

b. Show that the quadrilateral has two right angles.

Solution The slope of $\overline{AB} = \frac{-2 - (-4)}{6 - 2} = \frac{2}{4} = \frac{1}{2}$.



The slope of $\overline{BC} = \frac{6 - (-2)}{2 - 6} = \frac{8}{-4} = -2$.

The slope of $\overline{CD} = \frac{2 - 6}{-1 - 2} = \frac{-4}{-3} = \frac{4}{3}$.

The slope of $\overline{DA} = \frac{-4 - 2}{2 - (-1)} = \frac{-6}{3} = -2$.

a. $\overline{BC}$ and $\overline{DA}$ are parallel because they have equal slopes.

b. The slope of $\overline{AB}$ is the negative reciprocal of the slope of $\overline{BC}$, so they are perpendicular. Therefore, $\angle B$ is a right angle.

The slope of $\overline{AB}$ is the negative reciprocal of the slope of $\overline{DA}$, so they are perpendicular. Therefore, $\angle A$ is a right angle.

Answers **a.** $\overline{BC} \parallel \overline{DA}$ **b.** $\angle A$ and $\angle B$ are right angles.

EXAMPLE 2

Write an equation for l_1, the line through $(-2, 5)$ that is parallel to the line l_2 whose equation is $2x + y = 7$.

Solution (1) Solve the equation of l_2 for y:

$$2x + y = 7$$
$$y = -2x + 7$$

(2) Find the slope of l_2. The slope of a line in slope-intercept form is the coefficient of x:

$$y = \underbrace{-2}_{\text{slope}}x + 7$$

(3) Find the slope of l_1, which is equal to the slope of l_2:

$$\text{slope of } l_1 = -2$$

(4) Use the definition of slope to write an equation of l_1. Let (x, y) and $(-2, 5)$ be two points on l_1:

$$\frac{y_1 - y_2}{x_1 - x_2} = \text{slope}$$
$$\frac{y - 5}{x - (-2)} = -2$$
$$y - 5 = -2(x + 2)$$
$$y - 5 = -2x - 4$$
$$y = -2x + 1$$

Answer $y = -2x + 1$ or $2x + y = 1$

Exercises

Writing About Mathematics

1. If l_1 and l_2 have the same slope and have a common point, what must be true about l_1 and l_2?

2. Theorem 9.10 is true for all lines that are not vertical. Do vertical lines have the same slope? Explain your answer.

Developing Skills

In 3–8, for each pair of lines whose equations are given, tell whether the lines are parallel, perpendicular, or neither parallel nor perpendicular.

3. $x + y = 7$
$x - y = 3$

4. $2x - y = 5$
$y = 2x - 3$

5. $x = \frac{1}{3}y + 2$
$y = 3x - 2$

6. $2x + y = 6$
$2x - y = 3$

7. $x = 2$
$x = 5$

8. $x = 2$
$y = 3$

In 9–12, write an equation of the line that satisfies the given conditions.

9. Parallel to $y = -3x + 1$ with y-intercept 4.

10. Perpendicular to $y = -3x + 1$ with y-intercept 4.

11. Parallel to $x - 2y = 4$ and through the point $(4, 5)$.

12. Parallel to and 3 units below the x-axis.

Applying Skills

13. Quadrilateral $ABCD$ has two pairs of parallel sides, $\overline{AB} \parallel \overline{CD}$ and $\overline{BC} \parallel \overline{DA}$. The coordinates of A are $(1, 2)$, the coordinates of B are $(7, -1)$ and the coordinates of C are $(8, 2)$.

a. What is the slope of $\overline{AB}$?

b. What is the slope of $\overline{CD}$?

c. Write an equation for $\overleftrightarrow{CD}$.

d. What is the slope of $\overline{BC}$?

e. What is the slope of $\overline{AD}$?

f. Write an equation for $\overleftrightarrow{AD}$.

g. Use the equation of $\overleftrightarrow{CD}$ and the equation of $\overleftrightarrow{AD}$ to find the coordinates of D.

14. In quadrilateral $ABCD$, $\overline{BC} \perp \overline{AB}$, $\overline{DA} \perp \overline{AB}$, and $\overline{DA} \perp \overline{DC}$. The coordinates of A are $(1, -1)$, the coordinates of B are $(4, 2)$, and the coordinates of C are $(2, 4)$.

a. What is the slope of $\overline{AB}$?

b. What is the slope of $\overline{BC}$?

c. What is the slope of $\overline{AD}$? Justify your answer.

d. What is the slope of $\overline{DC}$? Justify your answer.

e. Write an equation for $\overleftrightarrow{DC}$.

f. Write an equation for $\overleftrightarrow{AD}$.

g. Use the equation of $\overleftrightarrow{DC}$ and the equation of $\overleftrightarrow{AD}$ to find the coordinates of D.

15. The coordinates of the vertices of quadrilateral $PQRS$ are $P(0, -1)$, $Q(4, 0)$, $R(2, 3)$, and $S(-2, 2)$.

a. Show that $PQRS$ has two pairs of parallel sides.

b. Show that $PQRS$ does not have a right angle.

16. The coordinates of the vertices of quadrilateral $KLMN$ are $K(-2, -1)$, $L(4, -3)$, $M(2, 1)$, and $N(-1, 2)$.

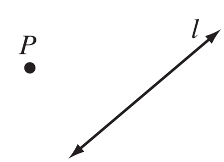
a. Show that $KLMN$ has only one pair of parallel sides.

b. Show that $KLMN$ has two right angles.

Hands-On Activity 1

In this activity, we will use a compass and a straightedge, or geometry software to construct a line parallel to a given line through a point not on the line.

STEP 1. Given a point, P, not on line, l. Through P, construct a line perpendicular to line l. Label this line n.



STEP 2. Through P, construct a line, p, perpendicular to line n.

Result: $l \parallel p$

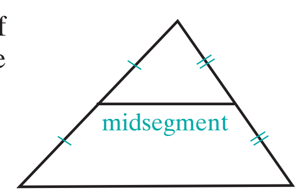
a. Justify the construction given in the procedure.

b. In (1)–(3), construct a line parallel to the line through the given point.

(1) $y = \frac{1}{4}x + 5; \left(3, 5\frac{1}{2}\right)$ (2) $-12x + y = 19; (12, -4)$ (3) $y = -\frac{1}{9}x - 3; (0, 4)$

Hands-On Activity 2

A **midsegment** is a segment formed by joining two midpoints of the sides of a triangle. In this activity, we will prove that a midsegment is parallel to the third side of the triangle using coordinate geometry.



1. With a partner or in a small group, complete the following:

a. Write the coordinates of a triangle using variables. These coordinates can be any convenient variables.

b. Find the midpoints of two sides of the triangle.

c. Prove that the midsegment formed is parallel to the third side of the triangle.

2. Compare your proof with the proofs of the other groups. Were different coordinates used? Which coordinates seem easiest to work with?

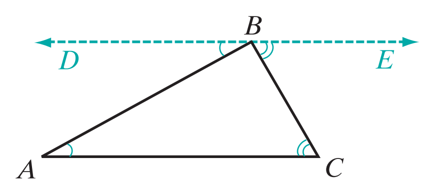
9-4 THE SUM OF THE MEASURES OF THE ANGLES OF A TRIANGLE

In previous courses, you have demonstrated that the sum of the measures of the angles of a triangle is 180 degrees. The congruent angles formed when parallel lines are cut by a transversal make it possible for us to prove this fact.

Theorem 9.11 The sum of the measures of the angles of a triangle is 180°.

Given $\triangle ABC$

Prove $m\angle A + m\angle B + m\angle C = 180$



Proof

Statements	Reasons
1. Let $\overleftrightarrow{DE}$ be the line through B that is parallel to $\overline{AC}$.	**1.** Through a given point not on a given line, there exists one and only one line parallel to the given line.
2. $m\angle DBE = 180$	**2.** A straight angle is an angle whose degree measure is 180.
3. $m\angle DBA + m\angle ABC + m\angle CBE = 180$	**3.** The whole is equal to the sum of all its parts.
4. $\angle A \cong \angle DBA$ and $\angle C \cong \angle CBE$	**4.** If two parallel lines are cut by a transversal, the alternate interior angles are congruent.
5. $m\angle A = m\angle DBA$ and $m\angle C = \angle CBE$	**5.** Congruent angles are equal in measure.
6. $m\angle A + m\angle ABC + m\angle C = 180$	**6.** Substitution postulate. ❑

Many corollaries to this important theorem exist.

Corollary 9.11a

> If two angles of one triangle are congruent to two angles of another triangle, then the third angles are congruent.

Proof: Let $\triangle ABC$ and $\triangle DEF$ be two triangles in which $\angle A \cong D$ and $\angle B \cong \angle E$. Since the sum of the degree measures of the angles of a triangle is 180, then

$$m\angle A + m\angle B + m\angle C = m\angle D + m\angle E + m\angle F.$$

We use the subtraction postulate to prove that $m\angle C = m\angle F$ and therefore, that $\angle C \cong \angle F$. ☐

Corollary 9.11b

> The acute angles of a right triangle are complementary.

Proof: In any triangle ABC, $m\angle A + m\angle B + m\angle C = 180$. If $\angle C$ is a right angle, $m\angle C = 90$,

$$m\angle A + m\angle B + 90 = 180$$
$$m\angle A + m\angle B = 90$$

Therefore, $\angle A$ and $\angle B$ are complementary. ☐

Corollary 9.11c

> Each acute angle of an isosceles right triangle measures 45°.

Proof: In isosceles right triangle ABC, $m\angle C = 90$ and $\overline{AC} \cong \overline{BC}$. Therefore, $m\angle A = m\angle B$. Using Corollary 9.12b, we know that $\angle A$ and $\angle B$ are complementary. Therefore, the measure of each must be 45. ☐

Corollary 9.11d

> Each angle of an equilateral triangle measures 60°.

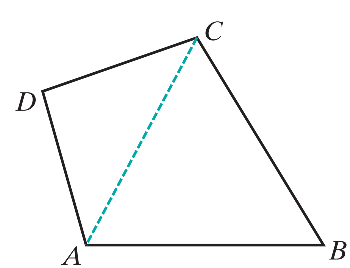
Proof: In equilateral triangle ABC, $m\angle A = m\angle B = m\angle C$. We substitute $m\angle A$ for $m\angle B$ and $m\angle C$ in the equation $m\angle A + m\angle B + m\angle C = 180$, and then solve the resulting equation: $3m\angle A = 180$ so $m\angle A = 60$. ☐

Corollary 9.11e

> The sum of the measures of the angles of a quadrilateral is 360°.

Proof: In quadrilateral $ABCD$, we draw $\overline{AC}$, forming two triangles. The sum of the measures of the angles of quadrilateral $ABCD$ is the sum of the measures of the angles of the two triangles:

$$180 + 180 = 360$$



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Corollary 9.11f

> The measure of an exterior angle of a triangle is equal to the sum of the measures of the nonadjacent interior angles.

The proof is left to the student. (See exercise 30.)

Note: Recall that the Exterior Angle Theorem of Section 7-5 gives an *inequality* that relates the exterior angle of a triangle to the nonadjacent interior angles: "The measure of an exterior angle of a triangle is *greater than* the measure of either nonadjacent interior angle." Corollary 9.11f is a version of the Exterior Angle Theorem involving *equality*.

EXAMPLE 1

The measure of the vertex angle of an isosceles triangle exceeds the measure of each base angle by 30 degrees. Find the degree measure of each angle of the triangle.

Solution Let x = measure of each base angle.

Let $x + 30$ = measure of vertex angle.

The sum of the measures of the angles of a triangle is 180.

$$\begin{aligned} x + x + x + 30 &= 180 \\ 3x + 30 &= 180 \\ 3x &= 150 \\ x &= 50 \\ x + 30 &= 80 \end{aligned}$$

Answer The angle measures are 50°, 50°, and 80°.

EXAMPLE 2

In $\triangle ABC$, the measures of the three angles are represented by $9x$, $3x - 6$, and $11x + 2$. Show that $\triangle ABC$ is a right triangle.

Solution Triangle ABC will be a right triangle if one of the angles is a right angle. Write and equation for the sum of the measures of the angles of $\triangle ABC$.

$$\begin{aligned} 9x + 3x - 6 + 11x + 2 &= 180 \\ 23x - 4 &= 180 \\ 23x &= 184 \\ x &= 8 \end{aligned}$$

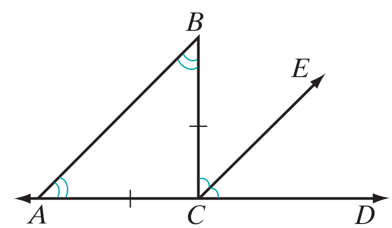
Substitute $x = 8$ in the representations of the angle measures.

$$9x = 9(8) = 72 \qquad 3x - 6 = 3(8) - 6 = 24 - 6 = 18 \qquad 11x + 2 = 11(8) + 2 = 88 + 2 = 90$$

Answer Triangle ABC is a right triangle because the degree measure of one of its angles is 90.

EXAMPLE 3

B is a not a point on $\overleftrightarrow{ACD}$. Ray $\overrightarrow{CE}$ bisects $\angle DCB$ and $\overline{AC} \cong \overline{BC}$. Prove that $\overleftrightarrow{AB} \parallel \overleftrightarrow{CE}$.



Solution *Given:* $\overrightarrow{CE}$ bisects $\angle DCB$ and $\overline{AC} \cong \overline{BC}$.

Prove: $\overleftrightarrow{AB} \parallel \overleftrightarrow{CE}$

Proof

Statements	Reasons
1. $\overrightarrow{CE}$ bisects $\angle DCB$.	**1.** Given.
2. $\angle DCE \cong \angle ECB$	**2.** Definition of an angle bisector.
3. m$\angle DCE$ = m$\angle ECB$	**3.** Measures of congruent angles are equal.
4. $\overline{AC} \cong \overline{BC}$	**4.** Given.
5. m$\angle CAB$ = m$\angle CBA$	**5.** Isosceles triangle theorem.
6. m$\angle DCB$ = m$\angle CAB$ + m$\angle CBA$	**6.** An exterior angle of a triangle is equal to the sum of the measures of the nonadjacent interior angles.
7. m$\angle DCB$ = m$\angle DCE$ + m$\angle ECB$	**7.** Partition postulate.
8. m$\angle DCE$ + m$\angle ECB$ = m$\angle CAB$ + m$\angle CBA$	**8.** Substitution postulate (steps 6 and 7).
9. m$\angle DCE$ + m$\angle DCE$ = m$\angle CAB$ + m$\angle CAB$ or 2m$\angle DCE$ = 2 m$\angle CAB$	**9.** Substitution postulate (steps 3, 5, and 8).
10. m$\angle DCE$ = m$\angle CAB$	**10.** Division postulate.
11. $\overleftrightarrow{AB} \parallel \overleftrightarrow{CE}$	**11.** If two lines are cut by a transversal forming equal corresponding angles, then the lines are parallel.

Exercises

Writing About Mathematics

1. McKenzie said that if a triangle is obtuse, two of the angles of the triangle are acute. Do you agree with McKenzie? Explain why or why not.

2. Giovanni said that since the sum of the measures of the angles of a triangle is 180, the angles of a triangle are supplementary. Do you agree with Giovanni? Explain why or why not.

Developing Skills

In 3–6, determine whether the given numbers can be the degree measures of the angles of a triangle.

3. 25, 100, 55 **4.** 95, 40, 45 **5.** 75, 75, 40 **6.** 12, 94, 74

In 7–10, the given numbers are the degree measures of two angles of a triangle. Find the measure of the third angle.

7. 80, 60 **8.** 45, 85 **9.** 90, 36 **10.** 65, 65

In 11–14, the measure of the vertex angle of an isosceles triangle is given. Find the measure of a base angle.

11. 20 **12.** 90 **13.** 76 **14.** 110

In 15–18, the measure of a base angle of an isosceles triangle is given. Find the measure of the vertex angle.

15. 80 **16.** 20 **17.** 45 **18.** 63

19. What is the measure of each exterior angle of an equilateral triangle?

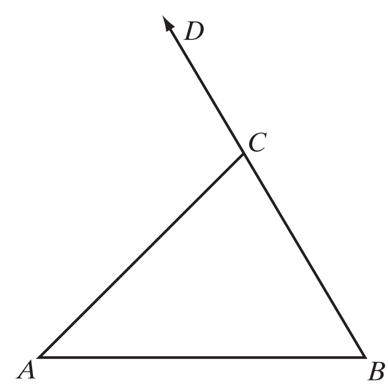
In 20–23, the diagram shows $\triangle ABC$ and exterior $\angle ACD$.

20. If $m\angle A = 40$ and $m\angle B = 20$, find $m\angle ACD$ and $m\angle ACB$.

21. If $m\angle A = 40$ and $m\angle B = 50$, find $m\angle ACD$ and $m\angle ACB$.

22. If $m\angle A = 40$ and $m\angle ACB = 120$, find $m\angle ACD$ and $m\angle B$.

23. If $m\angle A = 40$, $m\angle B = 3x + 20$, and $m\angle ACD = 5x + 10$, find $m\angle B$, $m\angle ACD$, and $m\angle ACB$.



Applying Skills

24. The measure of each base angle of an isosceles triangle is 21° more than the measure of the vertex angle. Find the measure of each angle of the triangle.

25. The measure of an exterior angle at C of isosceles $\triangle ABC$ is 110°. If $AC = BC$, find the measure of each angle of the triangle.

26. The measure of an exterior angle at D of isosceles $\triangle DEF$ is 100°. If $DE = EF$, find the measure of each angle of the triangle.

27. Triangle LMN is a right triangle with $\angle M$ the right angle. If $m\angle L = 32$, find the measure of $\angle N$ and the measure of the exterior angle at N.

28. In $\triangle ABC$, $m\angle A = 2x + 18$, $m\angle B = x + 40$, and $m\angle C = 3x - 40$.

a. Find the measure of each angle of the triangle.

b. Which is the longest side of the triangle?

29. The measure of an exterior angle at B, the vertex of isosceles $\triangle ABC$, can be represented by $3x + 12$. If the measure of a base angle is $2x - 2$, find the measure of the exterior angle and of the interior angles of $\triangle ABC$.

30. Prove Corollary 9.11f, "The measure of an exterior angle of a triangle is equal to the sum of the measures of the nonadjacent interior angles."

31. **a.** In the coordinate plane, graph points $A(5, 2)$, $B(2, 2)$, $C(2, -1)$, $D(-1, -1)$.

b. Draw $\overleftrightarrow{AB}$ and $\triangle BDC$.

c. Explain how you know that $\triangle BDC$ is an isosceles right triangle.

d. What is the measure of $\angle BDC$? Justify your answer.

e. What is the measure of $\angle DBA$? Justify your answer.

32. Prove that the sum of the measures of the angles of hexagon $ABCDEF$ is 720°. (*Hint:* draw $\overline{AD}$.)

33. $ABCD$ is a quadrilateral with $\overrightarrow{BD}$ the bisector of $\angle ABC$ and $\overrightarrow{DB}$ the bisector of $\angle ADC$. Prove that $\angle A \cong \angle C$.

9-5 PROVING TRIANGLES CONGRUENT BY ANGLE, ANGLE, SIDE

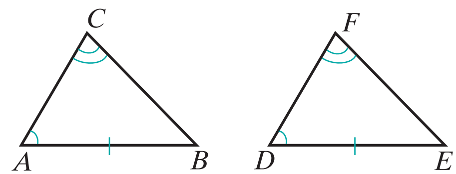
When two angles of one triangle are congruent to two angles of another triangle, the third angles are congruent. This is not enough to prove that the two triangles are congruent. We must know that at least one pair of corresponding sides are congruent. We already know that if two angles and the side between them in one triangle are congruent to the corresponding angles and side in another triangle, then the triangles are congruent by ASA. Now we want to prove angle-angle-side or **AAS triangle congruence**. This would allow us to conclude that if any two angles and any side in one triangle are congruent to the corresponding angles and side in another triangle, then the triangles are congruent.

Theorem 9.12

> If two angles and the side opposite one of them in one triangle are congruent to the corresponding angles and side in another triangle, then the triangles are congruent. (AAS)

Given $\triangle ABC$ and $\triangle DEF$, $\angle A \cong \angle D$, $\angle C \cong \angle F$, and $\overline{AB} \cong \overline{DE}$

Prove $\triangle ABC \cong \triangle DEF$



Proof

Statements	Reasons
1. $\angle A \cong \angle D$	**1.** Given.
2. $\angle C \cong \angle F$	**2.** Given.
3. $\angle B \cong \angle E$	**3.** If two angles of one triangle are congruent to two angles of another triangle, then the third angles are congruent.
4. $\overline{AB} \cong \overline{DE}$	**4.** Given.
5. $\triangle ABC \cong \triangle DEF$	**5.** ASA.

Therefore, when two angles and any side in one triangle are congruent to the corresponding two angles and side of a second triangle, we may say that the triangles are congruent either by ASA or by AAS.

The following corollaries can proved using AAS. Note that in every right triangle, the hypotenuse is the side opposite the right angle.

Corollary 9.12a

> Two right triangles are congruent if the hypotenuse and an acute angle of one right triangle are congruent to the hypotenuse and an acute angle of the other right triangle.

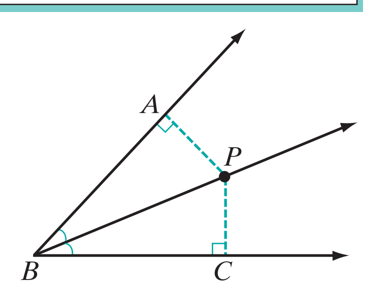
The proof uses AAS and is left to the student. (See exercise 15.)

Corollary 9.12b

> If a point lies on the bisector of an angle, then it is equidistant from the sides of the angle.

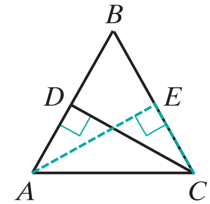
Recall that the distance from a point to a line is the length of the perpendicular from the point to the line. The proof uses AAS and is left to the student. (See exercise 16.)

You now have four ways to prove two triangles congruent: SAS, ASA, SSS, and AAS.



EXAMPLE 1

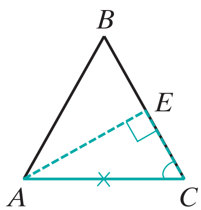
Prove that the altitudes drawn to the legs of an isosceles triangle from the endpoints of the base are congruent.

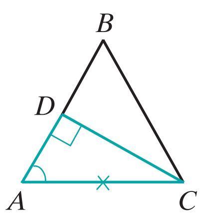


Given: Isosceles triangle ABC with $\overline{BA} \cong \overline{BC}$, $\overline{AE} \perp \overline{BC}$, and $\overline{CD} \perp \overline{BA}$.

Prove: $\overline{CD} \cong \overline{AE}$

Proof

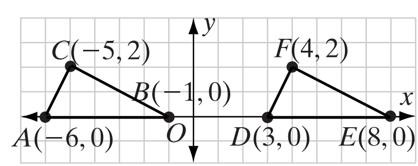




	Statements	Reasons
	1. In $\triangle ABC$, $\overline{BA} \cong \overline{BC}$.	**1.** Given.
A	**2.** $\angle BAC \cong \angle BCA$	**2.** If two sides of a triangle are congruent, the angles opposite these sides are congruent.
	3. $\overline{AE} \perp \overline{BC}$, $\overline{CD} \perp \overline{BA}$	**3.** Given.
	4. $\angle CDA$ and $\angle AEC$ are right angles.	**4.** Perpendicular lines are two lines that intersect to form right angles.
A	**5.** $\angle CDA \cong \angle AEC$	**5.** All right angles are congruent.
S	**6.** $\overline{AC} \cong \overline{AC}$	**6.** Reflexive property of congruence.
	7. $\triangle DAC \cong \triangle ECA$	**7.** AAS (steps 2, 5, and 6).
	8. $\overline{CD} \cong \overline{AE}$	**8.** Corresponding parts of congruent triangles are congruent.

EXAMPLE 2

The coordinates of the vertices of $\triangle ABC$ are $A(-6, 0)$, $B(-1, 0)$ and $C(-5, 2)$. The coordinates of $\triangle DEF$ are $D(3, 0)$, $E(8, 0)$, and $F(4, 2)$. Prove that the triangles are congruent.



Solution (1) Prove that the triangles are right triangles.

In $\triangle ABC$:	*In $\triangle DEF$:*
The slope of $\overline{AC}$ is $\frac{2-0}{-5-(-6)} = \frac{2}{1} = 2$.	The slope of $\overline{DF}$ is $\frac{2-0}{4-3} = \frac{2}{1} = 2$.
The slope of $\overline{CB}$ is $\frac{2-0}{-5-(-1)} = \frac{2}{-4} = -\frac{1}{2}$.	The slope of $\overline{FE}$ is $\frac{2-0}{4-8} = \frac{2}{-4} = -\frac{1}{2}$.

Two lines are perpendicular if the slope of one is the negative reciprocal of the slope of the other. Therefore, $\overline{AC} \perp \overline{CB}$, $\angle ACB$ is a right angle, and $\triangle ACB$ is a right triangle. Also, $\overline{DF} \perp \overline{FE}$, $\angle DFE$ is a right angle, and $\triangle DFE$ is a right triangle.

(2) Prove that two acute angles are congruent.
Two lines are parallel if their slopes are equal. Therefore, $\overline{CB} \parallel \overline{FE}$. The x-axis is a transversal forming congruent corresponding angles, so $\angle CBA$ and $\angle FED$ are congruent.

(3) Prove that the hypotenuses are congruent.
The hypotenuse of $\triangle ABC$ is $\overline{AB}$, and $AB = |-6 - (-1)| = 5$. The hypotenuse of $\triangle DEF$ is $\overline{DE}$, and $DE = |3 - 8| = 5$. Line segments that have the same measure are congruent, and so $\overline{AB} \cong \overline{DE}$.

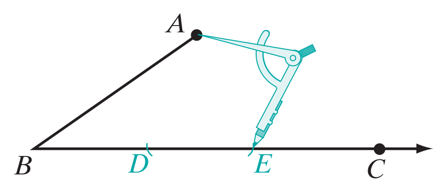
(4) Therefore, $\triangle ABC \cong \triangle DEF$ because the hypotenuse and an acute angle of one triangle are congruent to the hypotenuse and an acute angle of the other.

EXAMPLE 3

Show that if a triangle has two sides and an angle opposite one of the sides congruent to the corresponding sides and angle of another triangle, the triangles may not be congruent.

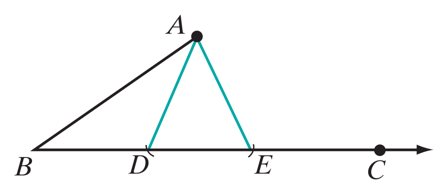
Solution (1) Draw an angle, $\angle ABC$.

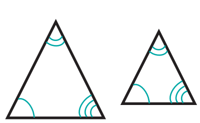
(2) Open a compass to a length that is smaller than AB but larger than the distance from A to $\overline{BC}$. Use the compass to mark two points, D and E, on $\overline{BC}$.



(3) Draw $\overline{AD}$ and $\overline{AE}$.

(4) In $\triangle ABD$ and $\triangle ABE$, $\overline{AB} \cong \overline{AB}$, $\angle B \cong \angle B$, and $\overline{AD} \cong \overline{AE}$. In these two triangles, two sides and the angle opposite one of the sides are congruent to the corresponding parts of the other triangle. But $\triangle ABD$ and $\triangle ABE$ are not congruent. This counterexample proves that SSA is not sufficient to prove triangles congruent.





Note: Triangles in which two sides and an angle opposite one of them are congruent *may not* be congruent to each other. Therefore, SSA is *not* a valid proof of triangle congruence. Similarly, triangles in which all three angles are congruent *may not* be congruent to each other, so AAA is also *not* a valid proof of triangle congruence.

Exercises

Writing About Mathematics

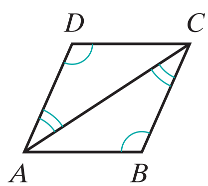
1. In Example 3, we showed that SSA cannot be used to prove two triangles congruent. Does this mean that whenever two sides and an angle opposite one of the sides are congruent to the corresponding parts of another triangle the two triangles are not congruent? Explain your answer.

2. In the coordinate plane, points A and C are on the same horizontal line and C and B are on the same vertical line. Are $\angle CAB$ and $\angle CBA$ complementary angles? Justify your answer.

Developing Skills

In 3–8, each figure shows two triangles. Congruent parts of the triangles have been marked. Tell whether or not the given congruent parts are sufficient to prove that the triangles are congruent. Give a reason for your answer.

3.



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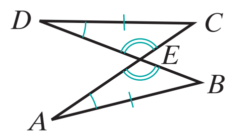

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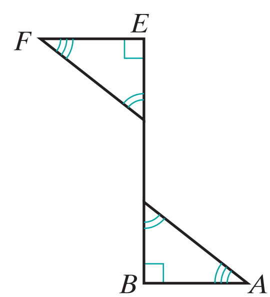

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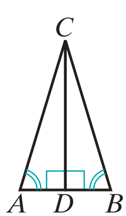

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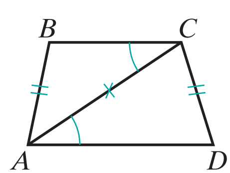

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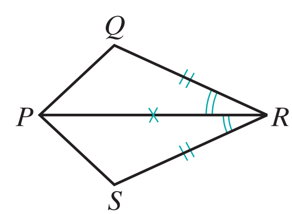


Applying Skills

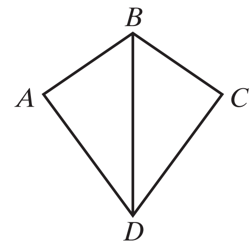
9. Prove that if two triangles are congruent, then the altitudes drawn from corresponding vertices are congruent.

10. Prove that if two triangles are congruent, then the medians drawn from corresponding vertices are congruent.

11. Prove that if two triangles are congruent, then the angle bisectors drawn from corresponding vertices are congruent.

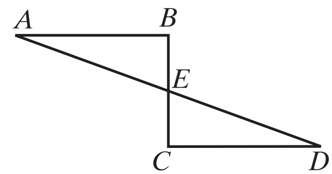
12. *Given:* Quadrilateral $ABCD$ with $\angle A \cong \angle C$ and $\overrightarrow{BD}$ the bisector of $\angle ABC$.

Prove: $\overrightarrow{DB}$ bisects $\angle ADC$.



13. *Given:* $\overline{AB} \parallel \overline{CD}$, $\overline{AB} \cong \overline{CD}$, and $\overline{AB} \perp \overline{BEC}$.

Prove: $\overline{AED}$ and $\overline{BEC}$ bisect each other.



14. **a.** Use a translation to prove that $\triangle ABC$ and $\triangle DEF$ in Example 2 are congruent.

b. Use two line reflections to prove that $\triangle ABC$ and $\triangle DEF$ in Example 2 are congruent.

15. Prove Corollary 9.12a, "Two right triangles are congruent if the hypotenuse and an acute angle of one right triangle are congruent to the hypotenuse and an acute angle of the other right triangle."

16. Prove Corollary 9.12b, "If a point lies on the bisector of an angle, it is equidistant from the sides of the angle."

17. Prove that if three angles of one triangle are congruent to the corresponding angles of another (AAA), the triangles may not be congruent. (Through any point on side $\overline{BC}$ of $\triangle ABC$, draw a line segment parallel to $\overline{AC}$.)

9-6 THE CONVERSE OF THE ISOSCELES TRIANGLE THEOREM

The Isosceles Triangle Theorem, proved in Section 5-3 of this book, is restated here in its conditional form.

► **If two sides of a triangle are congruent, then the angles opposite these sides are congruent.**

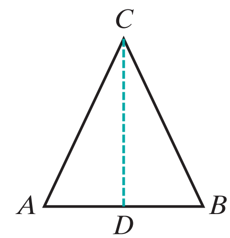
When we proved the Isosceles Triangle Theorem, its converse would have been very difficult to prove with the postulates and theorems that we had available at that time. Now that we can prove two triangles congruent by AAS, its converse is relatively easy to prove.

Theorem 9.13

If two angles of a triangle are congruent, then the sides opposite these angles are congruent.

Given $\triangle ABC$ with $\angle A \cong \angle B$.

Prove $\overline{CA} \cong \overline{CB}$



Proof We can use either the angle bisector or the altitude from C to separate the triangle into two congruent triangles. We will use the angle bisector.

Statements	Reasons
1. Draw $\overline{CD}$, the bisector of $\angle ACB$.	**1.** Every angle has one and only one bisector.
2. $\angle ACD \cong \angle BCD$	**2.** An angle bisector of a triangle is a line segment that bisects an angle of the triangle.
3. $\angle A \cong \angle B$	**3.** Given.
4. $\overline{CD} \cong \overline{CD}$	**4.** Reflexive property of congruence.
5. $\triangle ACD \cong \triangle BCD$	**5.** AAS.
6. $\overline{CA} \cong \overline{CB}$	**6.** Corresponding parts of congruent triangles are congruent.

The statement of the Isosceles Triangle Theorem (Theorem 5.1) and its converse (Theorem 9.14) can now be written in biconditional form:

► **Two angles of a triangle are congruent if and only if the sides opposite these angles are congruent.**

To prove that a triangle is isosceles, we may now prove that either of the following two statements is true:

1. Two sides of the triangle are congruent.
2. Two angles of the triangle are congruent.

Corollary 9.13a

If a triangle is equiangular, then it is equilateral.

Given $\triangle ABC$ with $\angle A \cong \angle B \cong \angle C$.

Prove $\triangle ABC$ is equilateral.

Proof We are given equiangular $\triangle ABC$. Then since $\angle A \cong \angle B$, the sides opposite these angles are congruent, that is, $\overline{BC} \cong \overline{AC}$. Also, since $\angle B \cong \angle C$, $\overline{AC} \cong \overline{AB}$ for the same reason. Therefore, $\overline{AC} \cong \overline{AB}$ by the transitive property of congruence, $\overline{AB} \cong \overline{BC} \cong \overline{CA}$, and $\triangle ABC$ is equilateral.

EXAMPLE 1

In $\triangle PQR$, $\angle Q \cong \angle R$. If $PQ = 6x - 7$ and $PR = 3x + 11$, find:

a. the value of x **b.** PQ **c.** PR

Solution **a.** Since two angles of $\triangle PQR$ are congruent, the sides opposite these angles are congruent. Thus, $PQ = PR$.

$$\begin{aligned} 6x - 7 &= 3x + 11 \\ 6x - 3x &= 11 + 7 \\ 3x &= 18 \\ x &= 6 \quad \textit{Answer} \end{aligned}$$

b.
$$\begin{aligned} PQ &= 6x - 7 \\ &= 6(6) - 7 \\ &= 36 - 7 \\ &= 29 \quad \textit{Answer} \end{aligned}$$

c.
$$\begin{aligned} PR &= 3x + 11 \\ &= 3(6) + 11 \\ &= 18 + 11 \\ &= 29 \quad \textit{Answer} \end{aligned}$$

EXAMPLE 2

The degree measures of the three angles of $\triangle ABC$ are represented by $m\angle A = x + 30$, $m\angle B = 3x$, and $m\angle C = 4x + 30$. Describe the triangle as acute, right, or obtuse, and as scalene, isosceles, or equilateral.

Solution The sum of the degree measures of the angles of a triangle is 180.

$$\begin{aligned} x + 30 + 3x + 4x + 30 &= 180 \\ 8x + 60 &= 180 \\ 8x &= 120 \\ x &= 15 \end{aligned}$$

Substitute $x = 15$ in the representations given for the three angle measures.

$$\begin{aligned} m\angle A &= x + 30 \\ &= 15 + 30 \\ &= 45 \end{aligned} \qquad \begin{aligned} m\angle B &= 3x \\ &= 3(15) \\ &= 45 \end{aligned} \qquad \begin{aligned} m\angle C &= 4x + 30 \\ &= 4(15) + 30 \\ &= 60 + 30 \\ &= 90 \end{aligned}$$

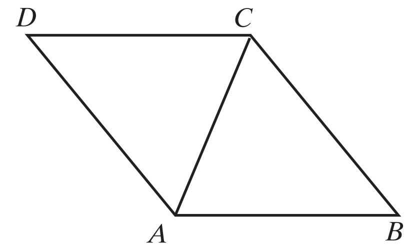
Since $\angle A$ and $\angle B$ each measure 45°, the triangle has two congruent angles and therefore two congruent sides. The triangle is isosceles. Also, since one angle measures 90°, the triangle is a right triangle.

Answer $\triangle ABC$ is an isosceles right triangle.

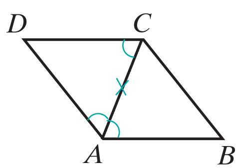
EXAMPLE 3

Given: Quadrilateral $ABCD$ with $\overline{AB} \parallel \overline{CD}$ and $\overrightarrow{AC}$ bisects $\angle DAB$.

Prove: $\overline{AD} \cong \overline{CD}$



Proof



Statements	Reasons
1. $\overline{AB} \parallel \overline{CD}$	**1.** Given.
2. $\angle DCA \cong \angle CAB$	**2.** If two parallel lines are cut by a transversal, the alternate interior angles are congruent.
3. $\overrightarrow{AC}$ bisects $\angle DAB$.	**3.** Given.
4. $\angle CAB \cong \angle DAC$	**4.** A bisector of an angle divides the angle into two congruent parts.
5. $\angle DCA \cong \angle DAC$ congruence.	**5.** Transitive property of
6. $\overline{AD} \cong \overline{CD}$	**6.** If two angles of a triangle are congruent, the sides opposite these angles are congruent.

Exercises

Writing About Mathematics

1. Julian said that the converse of the Isosceles Triangle Theorem could have been proved as a corollary to Theorem 7.3, "If the lengths of two sides of a triangle are unequal, then the measures of the angles opposite these sides are unequal." Do you agree with Julian? Explain why or why not.

2. Rosa said that if the measure of one angle of a right triangle is 45 degrees, then the triangle is an isosceles right triangle. Do you agree with Rosa? Explain why or why not.

In 3–6, in each case the degree measures of two angles of a triangle are given.

a. Find the degree measure of the third angle of the triangle.

b. Tell whether the triangle is isosceles or is not isosceles.

3. 70, 40 **4.** 30, 120

5. 50, 65 **6.** 80, 40

7. In $\triangle ABC$, $m\angle A = m\angle C$, $AB = 5x + 6$, and $BC = 3x + 14$. Find the value of x.

8. In $\triangle PQR$, $m\angle Q = m\angle P$, $PR = 3x$, and $RQ = 2x + 7$. Find PR and RQ.

9. In $\triangle MNR$, $MN = NR$, $m\angle M = 72$, and $m\angle R = 2x$. Find the measures of $\angle R$ and of $\angle N$.

10. In $\triangle ABC$, $m\angle A = 80$ and $m\angle B = 50$. If $AB = 4x - 4$, $AC = 2x + 16$, and $BC = 4x + 6$, find the measure of each side of the triangle.

11. The degree measures of the angles of $\triangle ABC$ are represented by $x + 10$, $2x$, and $2x - 30$. Show that $\triangle ABC$ is an isosceles triangle.

12. The degree measures of the angles of $\triangle ABC$ are represented by $x + 35$, $2x + 10$, and $3x - 15$. Show that $\triangle ABC$ is an equilateral triangle.

13. The degree measures of the angles of $\triangle ABC$ are represented by $3x + 18$, $4x + 9$, and $10x$. Show that $\triangle ABC$ is an isosceles right triangle.

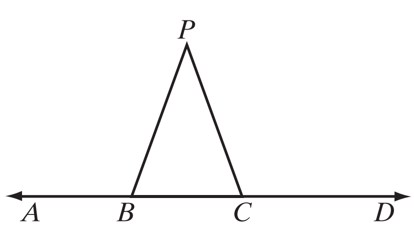
14. What is the measure of each exterior angle of an equilateral triangle?

15. What is the sum of the measures of the exterior angles of any triangle?

Applying Skills

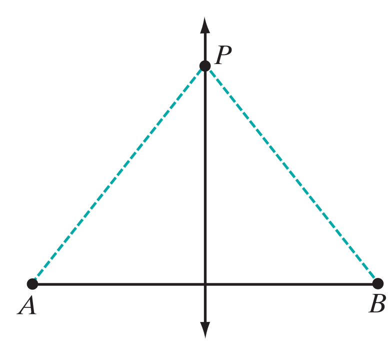
16. *Given:* P is not on $\overleftrightarrow{ABCD}$ and $\angle ABP \cong \angle PCD$.

Prove: $\triangle BPC$ is isosceles.



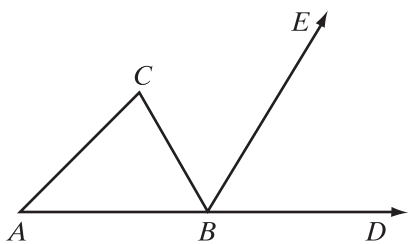
17. *Given:* P is not on $\overline{AB}$ and $\angle PAB \cong \angle PBA$.

Prove: P is on the perpendicular bisector of $\overline{AB}$.



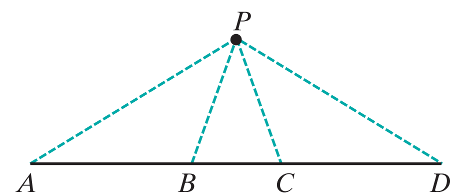
18. *Given:* $\overrightarrow{BE}$ bisects $\angle DBC$, an exterior angle of $\triangle ABC$, and $\overrightarrow{BE} \parallel \overline{AC}$.

Prove: $\overline{AB} \cong \overline{CB}$



19. *Given:* P is not on $\overline{ABCD}$, $\angle PBC \cong \angle PCB$, and $\angle APB \cong \angle DPC$

Prove: $\overline{AP} \cong \overline{DP}$



20. Prove Theorem 9.13 by drawing the altitude from C.

9-7 PROVING RIGHT TRIANGLES CONGRUENT BY HYPOTENUSE, LEG

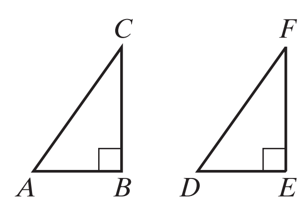
We showed in Section 5 of this chapter that, when two sides and an angle opposite one of these sides in one triangle are congruent to the corresponding two sides and angle in another triangle, the two triangles may or may not be congruent. When the triangles are right triangles, however, it is possible to prove that they are congruent. The congruent angles are the right angles, and each right angle is opposite the hypotenuse of the triangle.

Theorem 9.14

If the hypotenuse and a leg of one triangle are congruent to the corresponding parts of the other, then the two right triangles are congruent. (HL)

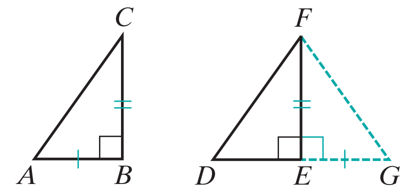
Given Right $\triangle ABC$ with right angle B and right $\triangle DEF$ with right angle E, $\overline{AC} \cong \overline{DF}$, $\overline{BC} \cong \overline{EF}$

Prove $\triangle ABC \cong \triangle DEF$



Proof To prove this theorem, we will construct a third triangle, $\triangle GEF$, that shares a common side with $\triangle DEF$ and prove that each of the two given triangles is congruent to $\triangle GEF$ and, thus, to each other.

We first show that $\triangle ABC$ is congruent to $\triangle GEF$:



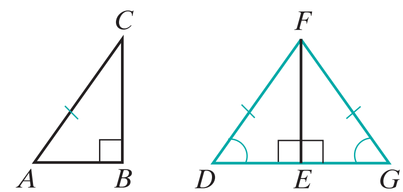
(1) Since any line segment may be extended any required length, extend $\overline{DE}$ to G so that $\overline{EG} \cong \overline{AB}$. Draw $\overline{FG}$.

(2) $\angle GEF$ and $\angle DEF$ form a linear pair, and $\angle DEF$ is a right angle. Therefore, $\angle GEF$ is a right angle. We are given that $\angle B$ is a right angle. All right angles are congruent, so $\angle B \cong \angle GEF$.

(3) We are also given $\overline{BC} \cong \overline{EF}$.

(4) Therefore, $\triangle ABC \cong \triangle GEF$ by SAS.

We now show that $\triangle DEF$ is also congruent to the constructed triangle, $\triangle GEF$:



(5) Since corresponding sides of congruent triangles are congruent, $\overline{AC} \cong \overline{GF}$. Since we are given $\overline{AC} \cong \overline{DF}$, $\overline{GF} \cong \overline{DF}$ by the transitive property of congruence.

(6) If two sides of a triangle are congruent, the angles opposite these sides are congruent. In $\triangle DFG$, $\overline{GF} \cong \overline{DF}$, so $\angle D \cong \angle G$. Also, $\angle DEF \cong \angle GEF$ since all right angles are congruent.

(7) Therefore, $\triangle DEF \cong \triangle GEF$ by AAS.

(8) Therefore, $\triangle ABC \cong \triangle DEF$ by the transitive property of congruence (steps 4 and 7).

This theorem is called the **hypotenuse-leg triangle congruence theorem**, abbreviated **HL**. Therefore, from this point on, when the hypotenuse and a leg of one right triangle are congruent to the corresponding parts of a second right triangle, we may say that the triangles are congruent.

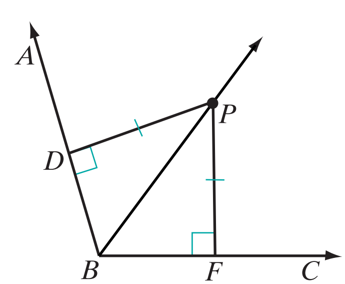
A corollary of this theorem is the converse of Corollary 9.12b.

Corollary 9.14a If a point is equidistant from the sides of an angle, then it lies on the bisector of the angle.

Given $\angle ABC$, $\overline{PD} \perp \overrightarrow{BA}$ at D, $\overline{PF} \perp \overrightarrow{BC}$ at F, and $PD = PF$

Prove $\angle ABP \cong \angle CBP$

Strategy Use HL to prove $\triangle PDB \cong \triangle PFB$.



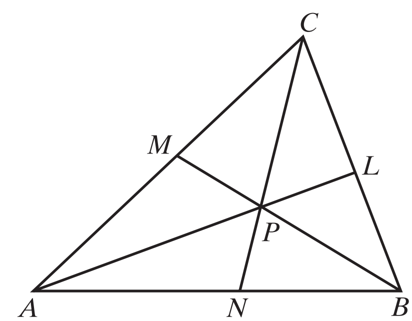
The proof of this theorem is left to the student. (See exercise 8.)

Concurrence of Angle Bisectors of a Triangle

In earlier chapters, we saw that the perpendicular bisectors of the sides of a triangle intersect in a point and that the altitudes of a triangle intersect in a point. Now we can prove that the angle bisectors of a triangle intersect in a point.

Theorem 9.15 The angle bisectors of a triangle are concurrent.

Given $\triangle ABC$ with $\overline{AL}$ the bisector of $\angle A$, $\overline{BM}$ the bisector of $\angle B$, and $\overline{CN}$ the bisector of $\angle C$.



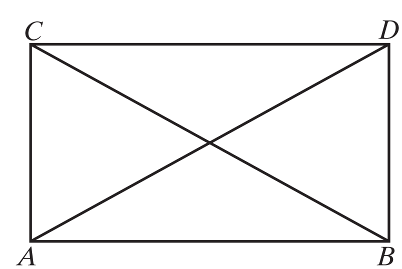
Prove $\overline{AL}$, $\overline{BM}$, and $\overline{CN}$ intersect in a point, P.

Proof Let P be the point at which $\overline{AL}$ and $\overline{BM}$ intersect. If a point lies on the bisector of an angle, then it is equidistant from the sides of the angle. Therefore, P is equidistant from $\overline{AC}$ and $\overline{AB}$ because it lies on the bisector of $\angle A$, and P is equidistant from $\overline{AB}$ and $\overline{BC}$ because it lies on the bisector of $\angle B$. Therefore, P is equidistant from $\overline{AC}$, $\overline{AB}$, and $\overline{BC}$. If a point is equidistant from the sides of an angle, then it lies on the bisector of the angle. Since P is equidistant from $\overline{AC}$ and $\overline{BC}$, then it lies of the bisector of $\angle C$. Therefore, the three angle bisectors of $\triangle ABC$ intersect at a point, P. ◻

The point where the angle bisectors of a triangle are concurrent is called the **incenter**.

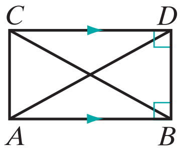
EXAMPLE 1

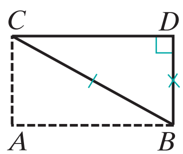
Given: $\triangle ABC$, $\overline{AB} \perp \overline{BD}$, $\overline{AB} \parallel \overline{DC}$, and $\overline{AD} \cong \overline{BC}$.

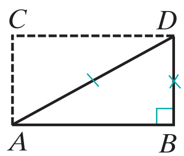


Prove: $\angle DAB \cong \angle BCD$

Proof We can show that $\triangle ADB$ and $\triangle CBD$ are right triangles and use HL to prove them congruent.







Statements	*Reasons*
1. $\overline{AB} \perp \overline{BD}$	**1.** Given.
2. $\overline{AB} \parallel \overline{DC}$	**2.** Given.
3. $\overline{BD} \perp \overline{DC}$	**3.** If a line is perpendicular to one of two parallel lines it is perpendicular to the other.
4. $\angle ABD$ and $\angle CDB$ are right angles.	**4.** Perpendicular lines intersect to form right angles.
5. $\overline{AD} \cong \overline{BC}$	**5.** Given.
6. $\overline{BD} \cong \overline{BD}$	**6.** Reflexive property of congruence.
7. $\triangle ADB \cong \triangle CBD$	**7.** HL (steps 5 and 6).
8. $\angle DAB \cong \angle BCD$	**8.** Corresponding parts of congruent triangles are congruent.

Exercises

Writing About Mathematics

1. In two right triangles, the right angles are congruent. What other pairs of corresponding parts must be known to be congruent in order to prove these two right triangles congruent?

2. The incenter of $\triangle ABC$ is P. If PD is the distance from P to $\overline{AB}$ and Q is any other point on $\overline{AB}$, is PD greater than PQ, equal to PQ, or less than PQ? Justify your answer.

Developing Skills

3. In $\triangle ABC$, $m\angle CAB = 40$ and $m\angle ABC = 60$. The angle bisectors of $\triangle ABC$ intersect at P.

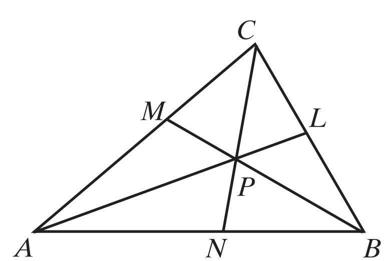
a. Find $m\angle BCA$.

b. Find the measure of each angle of $\triangle APB$.

c. Find the measure of each angle of $\triangle BPC$.

d. Find the measure of each angle of $\triangle CPA$.

e. Does the bisector of $\angle CAB$ also bisect $\angle CPB$? Explain your answer.



4. Triangle ABC is an isosceles right triangle with the right angle at C. Let P be the incenter of $\triangle ABC$.

a. Find the measure of each acute angle of $\triangle ABC$.

b. Find the measure of each angle of $\triangle APB$.

c. Find the measure of each angle of $\triangle BPC$.

d. Find the measure of each angle of $\triangle CPA$.

e. Does the bisector of $\angle ACB$ also bisect $\angle APB$? Explain your answer.

5. Triangle ABC is an isosceles triangle with $m\angle C = 140$. Let P be the incenter of $\triangle ABC$.

a. Find the measure of each acute angle of $\triangle ABC$.

b. Find the measure of each angle of $\triangle APB$.

c. Find the measure of each angle of $\triangle BPC$.

d. Find the measure of each angle of $\triangle CPA$.

e. Does the bisector of $\angle ACB$ also bisect $\angle APB$? Explain your answer.

6. In $\triangle RST$, the angle bisectors intersect at P. If $m\angle RTS = 50$, $m\angle TPR = 120$, and $m\angle RPS = 115$, find the measures of $\angle TRS$, $\angle RST$, and $\angle SPT$.

7. a. Draw a scalene triangle on a piece of paper or using geometry software. Label the triangle $\triangle ABC$.

b. Using compass and straightedge or geometry software, construct the angle bisectors of the angles of the triangle. Let $\overline{AL}$ be the bisector of $\angle A$, $\overline{BM}$ be the bisector of $\angle B$, and $\overline{CN}$ be the bisector of $\angle C$, such that L, M, and N are points on the triangle.

c. Label the incenter P.

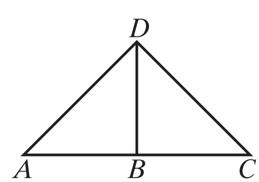
d. In $\triangle ABC$, does $AP = BP = CP$? Explain why or why not.

e. If the incenter is equidistant from the vertices of $\triangle DEF$, what kind of a triangle is $\triangle DEF$?

Applying Skills

8. Prove Corollary 9.14a, "If a point is equidistant from the sides of an angle, then it lies on the bisector of the angle."

9. Given $\overline{DB} \perp \overline{ABC}$ and $\overline{AD} \perp \overline{DC}$, when is $\triangle ABD$ congruent to $\triangle DBC$? Explain.



10. When we proved that the bisectors of the angles of a triangle intersect in a point, we began by stating that two of the angle bisectors, $\overline{AL}$ and $\overline{BM}$, intersect at P. To prove that they intersect, show that they are not parallel. (*Hint:* $\overline{AL}$ and $\overline{BM}$ are cut by transversal $\overline{AB}$. Show that a pair of interior angles on the same side of the transversal cannot be supplementary.)

11. *Given:* Quadrilateral $ABCD$, $\overline{AB} \perp \overline{BD}$, $\overline{BD} \perp \overline{DC}$, and $\overline{AD} \cong \overline{CB}$.

Prove: $\angle A \cong \angle C$ and $\overline{AD} \parallel \overline{CB}$

12. In $\triangle QRS$, the bisector of $\angle QRS$ is perpendicular to $\overline{QS}$ at P.

a. Prove that $\triangle QRS$ is isosceles.

b. Prove that P is the midpoint of $\overline{QS}$.

13. Each of two lines from the midpoint of the base of an isosceles triangle is perpendicular to one of the legs of the triangle. Prove that these lines are congruent.

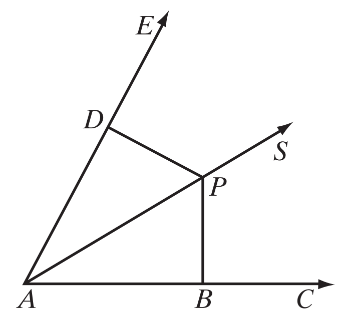
14. In quadrilateral $ABCD$, $\angle A$ and $\angle C$ are right angles and $AB = CD$. Prove that:

a. $AD = BC$ **b.** $\angle ABD \cong \angle CDB$ **c.** $\angle ADC$ is a right angle.

15. In quadrilateral $ABCD$, $\angle ABC$ and $\angle BCD$ are right angles, and $AC = BD$. Prove that $AB = CD$.

16. *Given:* $\overrightarrow{ABC}$, $\overrightarrow{APS}$, and $\overrightarrow{ADE}$ with $\overline{PB} \perp \overrightarrow{ABC}$, $\overline{PD} \perp \overrightarrow{ADE}$, and $PB = PD$.

Prove: $\overrightarrow{APS}$ bisects $\angle CAE$.



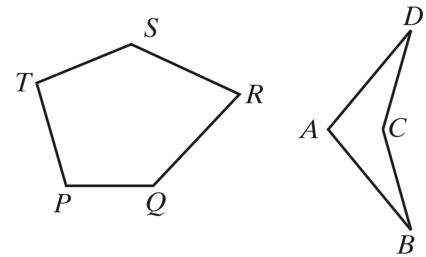
9-8 INTERIOR AND EXTERIOR ANGLES OF POLYGONS

Polygons

Recall that a polygon is a closed figure that is the union of line segments in a plane. Each vertex of a polygon is the endpoint of two line segments. We have proved many theorems about triangles and have used what we know about triangles to prove statements about the sides and angles of quadrilaterals, polygons with four sides. Other common polygons are:

- A **pentagon** is a polygon that is the union of five line segments.
- A **hexagon** is a polygon that is the union of six line segments.
- An **octagon** is a polygon that is the union of eight line segments.
- A **decagon** is a polygon that is the union of ten line segments.
- In general, an ***n*-gon** is a polygon with n sides.

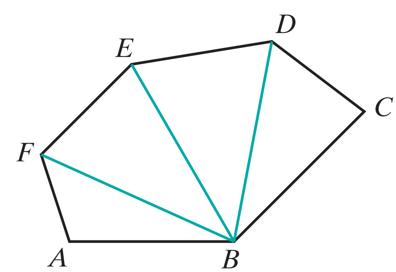
A **convex polygon** is a polygon in which each of the interior angles measures less than 180 degrees. Polygon $PQRST$ is a convex polygon and a pentagon. A **concave polygon** is a polygon in which at least one interior angle measures more than 180 degrees. Polygon $ABCD$ is a concave polygon and a quadrilateral. In the rest of this textbook, unless otherwise stated, all polygons are convex.

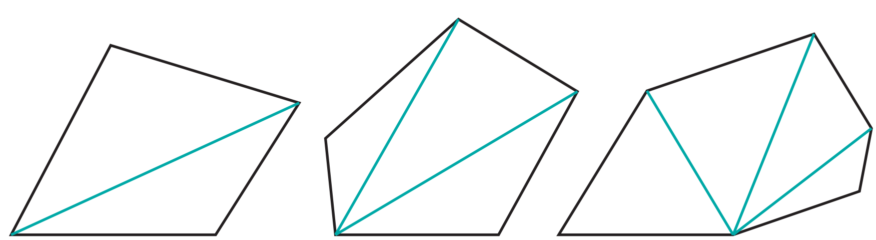


Interior Angles of a Polygon

A pair of angles whose vertices are the endpoints of a common side are called **consecutive angles**. And the vertices of consecutive angles are called **consecutive vertices** or **adjacent vertices**. For example, in $PQRST$, $\angle P$ and $\angle Q$ are consecutive angles and P and Q are consecutive or adjacent vertices. Another pair of consecutive angles are $\angle T$ and $\angle P$. Vertices R and T are nonadjacent vertices.

A **diagonal** of a polygon is a line segment whose endpoints are two nonadjacent vertices. In hexagon $ABCDEF$, the vertices adjacent to B are A and C and the vertices nonadjacent to B are D, E, and F. Therefore, there are three diagonals with endpoint B: $\overline{BD}$, $\overline{BE}$, and $\overline{BF}$.





The polygons shown above have four, five, and six sides. In each polygon, all possible diagonals from a vertex are drawn. In the quadrilateral, two triangles are formed. In the pentagon, three triangles are formed, and in the hexagon, four triangles are formed. Note that in each polygon, the number of triangles formed is two less than the number of sides.

- In a quadrilateral: the sum of the measures of the angles is $2(180) = 360$.
- In a pentagon: the sum of the measures of the angles is $3(180) = 540$.
- In a hexagon: the sum of the measures of the angles is $4(180) = 720$.

In general, the number of triangles into which the diagonals from a vertex separate a polygon of n sides is two less than the number of sides, or $n - 2$. The sum of the interior angles of the polygon is the sum of the interior angles of the triangles formed, or $180(n - 2)$. We have just proved the following theorem:

Theorem 9.16

> The sum of the measures of the interior angles of a polygon of n sides is $180(n - 2)°$.

Exterior Angles of a Polygon

At any vertex of a polygon, an exterior angle forms a linear pair with the interior angle. The interior angle and the exterior angle are supplementary. Therefore, the sum of their measures is 180°. If a polygon has n sides, the sum of the interior and exterior angles of the polygon is $180n$. Therefore, in a polygon with n sides:

$$\begin{aligned}\text{The measures of the exterior angles} &= 180n - \text{the measures of the interior angles}\\ &= 180n - 180(n - 2)\\ &= 180n - 180n + 360\\ &= 360\end{aligned}$$

We have just proved the following theorem:

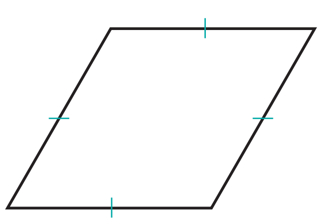
Theorem 9.17

> The sum of the measures of the exterior angles of a polygon is 360°.

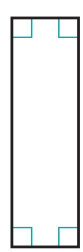
DEFINITION

A **regular polygon** is a polygon that is both equilateral and equiangular.

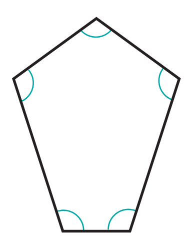
If a triangle is equilateral, then it is equiangular. For polygons that have more than three sides, the polygon can be equiangular and not be equilateral, or can be equilateral and not be equiangular.



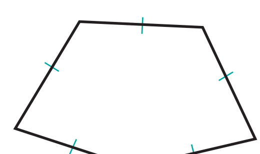
Equilateral but not equiangular



Equiangular but not equilateral



Equiangular but not equilateral



Equilateral but not equiangular

EXAMPLE 1

The measure of an exterior angle of a regular polygon is 45 degrees.

a. Find the number of sides of the polygon.

b. Find the measure of each interior angle.

c. Find the sum of the measures of the interior angles.

Solution **a.** Let n be the number of sides of the polygon. Then the sum of the measures of the exterior angles is n times the measure of one exterior angle.

$$45n = 360$$
$$n = \tfrac{360}{45}$$
$$n = 8 \quad \textit{Answer}$$

b. Each interior angle is the supplement of each exterior angle.

$$\begin{aligned}\text{Measure of each interior angle} &= 180 - 45\\ &= 135 \quad \textit{Answer}\end{aligned}$$

c. Use the sum of the measures of the interior angles, $180(n - 2)$.

$$\begin{aligned}180(n-2) &= 180(8-2)\\ &= 180(6)\\ &= 1{,}080 \quad \textit{Answer}\end{aligned}$$

or

Multiply the measure of each interior angle by the number of sides.

$$8(135) = 1{,}080 \quad \textit{Answer}$$

Answers **a.** 8 sides **b.** 135° **c.** 1,080°

EXAMPLE 2

In quadrilateral $ABCD$, $m\angle A = x$, $m\angle B = 2x - 12$, $m\angle C = x + 22$, and $m\angle D = 3x$.

a. Find the measure of each interior angle of the quadrilateral.

b. Find the measure of each exterior angle of the quadrilateral.

Solution **a.**

$$\begin{aligned}m\angle A + m\angle B + m\angle C + m\angle D &= 18(n-2)\\ x + 2x - 12 + x + 22 + 3x &= 180(4-2)\\ 7x + 10 &= 360\\ 7x &= 350\\ x &= 50\end{aligned}$$

$$\begin{aligned} m\angle A &= x \\ &= 50 \end{aligned} \qquad \begin{aligned} m\angle B &= 2x - 12 \\ &= 2(50) - 12 \\ &= 88 \end{aligned}$$

$$\begin{aligned} m\angle C &= x + 22 \\ &= 50 + 22 \\ &= 72 \end{aligned} \qquad \begin{aligned} m\angle D &= 3x \\ &= 3(50) \\ &= 150 \end{aligned}$$

b. Each exterior angle is the supplement of the interior angle with the same vertex.

The measure of the exterior angle at A is $180 - 50 = 130$.

The measure of the exterior angle at B is $180 - 88 = 92$.

The measure of the exterior angle at C is $180 - 72 = 108$.

The measure of the exterior angle at D is $180 - 150 = 30$.

Answers **a.** 50°, 88°, 72°, 150° **b.** 130°, 92°, 108°, 30°

Exercises

Writing About Mathematics

1. Taylor said that each vertex of a polygon with n sides is the endpoint of $(n - 3)$ diagonals. Do you agree with Taylor? Justify your answer.

2. Ryan said that every polygon with n sides has $\frac{n}{2}(n - 3)$ diagonals. Do you agree with Ryan? Justify your answer.

Developing Skills

3. Find the sum of the degree measures of the interior angles of a polygon that has:

a. 3 sides **b.** 7 sides **c.** 9 sides **d.** 12 sides

4. Find the sum of the degree measures of the interior angles of:

a. a hexagon **b.** an octagon **c.** a pentagon **d.** a quadrilateral

5. Find the sum of the measures of the exterior angles of a polygon that has:

a. 4 sides **b.** 8 sides **c.** 10 sides **d.** 36 sides

In 6–14, for each *regular* polygon with the given number of sides, find the degree measures of: **a.** one exterior angle **b.** one interior angle

6. 4 sides **7.** 5 sides **8.** 6 sides

9. 8 sides **10.** 9 sides **11.** 12 sides

12. 20 sides **13.** 36 sides **14.** 42 sides

15. Find the number of sides of a *regular* polygon each of whose exterior angles contains:

a. 30° **b.** 45° **c.** 60° **d.** 120°

16. Find the number of sides of a *regular* polygon each of whose interior angles contains:

a. 90° **b.** 120° **c.** 140° **d.** 160°

17. Find the number of sides a polygon if the sum of the degree measures of its interior angles is:

a. 180 **b.** 360 **c.** 540 **d.** 900

e. 1,440 **f.** 2,700 **g.** 1,800 **h.** 3,600

Applying Skills

18. The measure of each interior angle of a regular polygon is three times the measure of each exterior angle. How many sides does the polygon have?

19. The measure of each interior angle of a regular polygon is 20 degrees more than three times the measure of each exterior angle. How many sides does the polygon have?

20. The sum of the measures of the interior angles of a *concave* polygon is also $180(n - 2)$, where n is the number of sides. Is it possible for a concave quadrilateral to have two interior angles that are both more than 180°? Explain why or why not.

21. From vertex A of regular pentagon $ABCDE$, two diagonals are drawn, forming three triangles.

a. Prove that two of the triangles formed by the diagonals are congruent.

b. Prove that the congruent triangles are isosceles.

c. Prove that the third triangle is isosceles.

22. From vertex L of regular hexagon $LMNRST$, three diagonals are drawn, forming four triangles.

a. Prove that two of the triangles formed by the diagonals are congruent.

b. Prove that the other two triangles formed by the diagonals are congruent.

c. Find the measures of each of the angles in each of the four triangles.

23. The coordinates of the vertices of quadrilateral $ABCD$ are $A(-2, 0)$, $B(0, -2)$, $C(2, 0)$, and $D(0, 2)$.

a. Prove that each angle of the quadrilateral is a right angle.

b. Segments of the x-axis and the y-axis are diagonals of the quadrilateral. Prove that the four triangles into which the diagonals separate the quadrilateral are congruent.

c. Prove that $ABCD$ is a regular quadrilateral.

Hands-On Activity

In Section 9-7, we saw that the angle bisectors of a triangle are concurrent in a point called the *incenter*. In this activity, we will study the intersection of the angle bisectors of polygons.

a. Draw various polygons that are *not* regular of different sizes and numbers of sides. Construct the angle bisector of each interior angle. Do the angle bisectors appear to intersect in a single point?

b. Draw various *regular* polygons of different sizes and numbers of sides. Construct the angle bisector of each interior angle. Do the angle bisectors appear to intersect in a single point?

c. Based on the results of part **a** and **b**, state a conjecture regarding the intersection of the angle bisector of polygons.

CHAPTER SUMMARY

Definitions to Know

- **Parallel lines** are coplanar lines that have no points in common, or have all points in common and, therefore, coincide.
- A **transversal** is a line that intersects two other coplanar lines in two different points.
- The **incenter** is the point of intersection of the bisectors of the angles of a triangle.
- A **convex polygon** is a polygon in which each of the interior angles measures less than 180 degrees.
- A **concave polygon** is a polygon in which at least one of the interior angles measures more than 180 degrees.
- A **regular polygon** is a polygon that is both equilateral and equiangular.

Postulates

9.1 Two distinct coplanar lines are either parallel or intersecting.

9.2 Through a given point not on a given line, there exists one and only one line parallel to the given line.

Theorems and Corollaries

9.1 Two coplanar lines cut by a transversal are parallel if and only if the alternate interior angles formed are congruent.

9.2 Two coplanar lines cut by a transversal are parallel if and only if corresponding angles are congruent.

9.3 Two coplanar lines cut by a transversal are parallel if and only if interior angles on the same side of the transversal are supplementary.

9.4 If two coplanar lines are each perpendicular to the same line, then they are parallel.

9.5 If, in a plane, a line intersects one of two parallel lines, it intersects the other.

9.6 If a transversal is perpendicular to one of two parallel lines, it is perpendicular to the other.

9.7 If two of three lines in the same plane are each parallel to the third line, then they are parallel to each other.

9.8 If two lines are vertical lines, then they are parallel.

9.9 If two lines are horizontal lines, then they are parallel.

9.10 Two non-vertical lines in the coordinate plane are parallel if and only if they have the same slope.

9.11 The sum of the measures of the angles of a triangle is 180°.

9.11a If two angles of one triangle are congruent to two angles of another triangle, then the third angles are congruent.

9.11b The acute angles of a right triangle are complementary.

9.11c Each acute angle of an isosceles right triangle measures 45°.

9.11d Each angle of an equilateral triangle measures 60°.

9.11e The sum of the measures of the angles of a quadrilateral is 360°.

9.11f The measure of an exterior angle of a triangle is equal to the sum of the measures of the nonadjacent interior angles.

9.12 If two angles and the side opposite one of them in one triangle are congruent to the corresponding angles and side in another triangle, then the triangles are congruent. (AAS)

9.12a Two right triangles are congruent if the hypotenuse and an acute angle of one right triangle are congruent to the hypotenuse and an acute angle of the other right triangle.

9.12b If a point lies on the bisector of an angle, then it is equidistant from the sides of the angle.

9.13 If two angles of a triangle are congruent, then the sides opposite these angles are congruent.

9.13a If a triangle is equiangular, then it is equilateral.

9.14 If the hypotenuse and a leg of one right triangle are congruent to the corresponding parts of the other, then the two right triangles are congruent. (HL)

9.14a If a point is equidistant from the sides of an angle, then it lies on the bisector of the angle.

9.15 The angle bisectors of a triangle are concurrent.

9.16 The sum of the measures of the interior angles of a polygon of n sides is $180(n - 2)°$.

9.17 The sum of the measures of the exterior angles of a polygon is 360°.

VOCABULARY

9-1 Euclid's parallel postulate • Playfair's postulate • Coplanar • Parallel lines • Transversal • Interior angles • Exterior angles • Alternate interior angles • Alternate exterior angles • Interior angles on the same side of the transversal • Corresponding angles

9-3 Midsegment

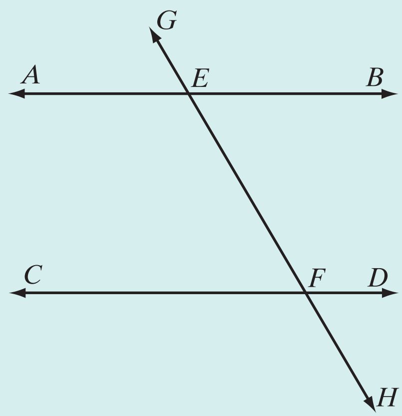
9-5 AAS triangle congruence

9-7 Hypotenuse-leg triangle congruence theorem (HL) • Incenter

9-8 Pentagon • Hexagon • Octagon • Decagon • n-gon • Convex polygon • Concave polygon • Consecutive angles • Consecutive vertices • Adjacent vertices • Diagonal of a polygon • Regular polygon

REVIEW EXERCISES

In 1–5, $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ and these lines are cut by transversal $\overleftrightarrow{GH}$ at points E and F, respectively.



1. If m$\angle AEF = 5x$ and m$\angle DFE = 75$, find x.

2. If m$\angle CFE = 3y + 20$ and m$\angle AEG = 4y - 10$, find y.

3. If m$\angle BEF = 5x$ and m$\angle CFE = 7x - 48$, find x.

4. If m$\angle DFE = y$ and m$\angle BEF = 3y - 40$, find m$\angle DFE$.

5. If m$\angle AEF = 4x$ and m$\angle EFD = 3x + 18$, find:

a. the value of x **b.** m$\angle AEF$

c. m$\angle EFD$ **d.** m$\angle BEF$ **e.** m$\angle CFH$

6. The degree measure of the vertex angle of an isosceles triangle is 120. Find the measure of a base angle of the triangle.

7. In $\triangle ABC$, $\angle A \cong \angle C$. If $AB = 8x + 4$ and $CB = 3x + 34$, find x.

8. In an isosceles triangle, if the measure of the vertex angle is 3 times the measure of a base angle, find the degree measure of a base angle.

9. In a triangle, the degree measures of the three angles are represented by x, $x + 42$, and $x - 6$. Find the angle measures.

10. In $\triangle PQR$, if m$\angle P = 35$ and m$\angle Q = 85$, what is the degree measure of an exterior angle of the triangle at vertex R?

11. An exterior angle at the base of an isosceles triangle measures 130°. Find the measure of the vertex angle.

12. In $\triangle ABC$, if $\overline{AB} \cong \overline{AC}$ and m$\angle A = 70$, find m$\angle B$.

13. In $\triangle DEF$, if $\overline{DE} \cong \overline{DF}$ and m$\angle E = 13$, find m$\angle D$.

14. In $\triangle PQR$, $\overline{PQ}$ is extended through Q to point T, forming exterior $\angle RQT$. If m$\angle RQT = 70$ and m$\angle R = 10$, find m$\angle P$.

15. In $\triangle ABC$, $\overline{AC} \cong \overline{BC}$. The degree measure of an exterior angle at vertex C is represented by $5x + 10$. If m$\angle A = 30$, find x.

16. The degree measures of the angles of a triangle are represented by $x - 10$, $2x + 20$, and $3x - 10$. Find the measure of each angle of the triangle.

17. If the degree measures of the angles of a triangle are represented by x, y, and $x + y$, what is the measure of the largest angle of the triangle?

18. If parallel lines are cut by a transversal so that the degree measures of two corresponding angles are represented by $2x + 50$ and $3x + 20$, what is the value of x?

19. The measure of one exterior angle of a regular polygon is 30°. How many sides does the regular polygon have?

20. What is the sum of the degree measures of the interior angles of a polygon with nine sides?

21. *Given:* Right triangle ABC with $\angle C$ the right angle.

Prove: $AB > AC$

22. *Given:* $\overline{AEB}$ and $\overline{CED}$ bisect each other at E.

Prove: $\overline{AC} \parallel \overline{BD}$

23. P is not on $\overline{ABCD}$ and $\overline{PA}$, $\overline{PB}$, $\overline{PC}$, and $\overline{PD}$ are drawn. If $\overline{PB} \cong \overline{PC}$ and $\angle APB \cong \angle DPC$, prove that $\overline{PA} \cong \overline{PD}$.

24. P is not on $\overline{ABCD}$ and $\overline{PA}$, $\overline{PB}$, $\overline{PC}$, and $\overline{PD}$ are drawn. If $\angle PBC \cong \angle PCB$ and $\overline{AB} \cong \overline{DC}$, prove that $\overline{PA} \cong \overline{PD}$.

25. Herbie wanted to draw pentagon $ABCDE$ with $m\angle A = m\angle B = 120$ and $m\angle C = m\angle D = 150$. Is such a pentagon possible? Explain your answer.

Exploration

The geometry that you have been studying is called plane Euclidean geometry. Investigate a non-Euclidean geometry. How do the postulates of a non-Euclidean geometry differ from the postulates of Euclid? How can the postulates from this chapter be rewritten to fit the non-Euclidean geometry you investigated? What theorems from this chapter are not valid in the non-Euclidean geometry that you investigated? One possible non-Euclidean geometry is the geometry of the sphere suggested in the Chapter 1 Exploration.

CUMULATIVE REVIEW — Chapters 1–9

Part I

Answer all questions in this part. Each correct answer will receive 2 credits. No partial credit will be allowed.

1. If M is the midpoint of $\overline{AB}$, which of the following may be false?

(1) M is between A and B.

(2) $AM = MB$

(3) A, B, and M are collinear.

(4) $\overleftrightarrow{MN}$, a line that intersects $\overline{AB}$ at M, is the perpendicular bisector of $\overline{AB}$.

2. The statement "If two angles form a linear pair, then they are supplementary" is true. Which of the following statements must also be true?

(1) If two angles do not form a linear pair, then they are not supplementary.

(2) If two angles are not supplementary, then they do not form a linear pair.

(3) If two angles are supplementary, then they form a linear pair.

(4) Two angles form a linear pair if and only if they are supplementary.

3. Which of the following is a statement of the reflexive property of equality for all real numbers a, b, and c?

(1) $a = a$

(2) If $a = b$, then $b = a$.

(3) If $a = b$ and $b = c$, then $a = c$.

(4) If $a = b$, then $ac = bc$.

4. Two angles are complementary. If the measure of the larger angle is 10 degrees less than three times the measure of the smaller, what is the measure of the larger angle?

(1) 20° (2) 25° (3) 65° (4) 70°

5. Under the transformation $r_{x\text{-axis}} \circ R_{90°}$, the image of $(-2, 5)$ is

(1) $(-5, -2)$ (2) $(-5, 2)$ (3) $(5, -2)$ (4) $(2, -5)$

6. An equation of the line through $(0, -1)$ and perpendicular to the line $x + 3y = 4$ is

(1) $3x + y = 1$

(2) $x - 3y = 1$

(3) $3x - y = 1$

(4) $x + 3y = -1$

7. The coordinates of the midpoint of the line segment whose endpoints are $(-3, 4)$ and $(5, -6)$ are

(1) $(1, -1)$ (2) $(-4, 5)$ (3) $(4, 5)$ (4) $(-4, -5)$

8. If a, b, c, and d are real numbers and $a > b$ and $c > d$, which of the following must be true?

(1) $a + c > b + d$

(2) $a - c > b - d$

(3) $ac > bc$

(4) $\frac{a}{c} > \frac{b}{d}$

9. The measure of each base angle of an isosceles triangle is 5 more than twice the measure of the vertex angle. The measure of the vertex angle is

(1) 34° (2) 73° (3) 43.75° (4) 136.25°

10. Which of the following properties is not preserved under a line reflection?

(1) distance

(2) orientation

(3) angle measure

(4) midpoint

Part II

Answer all questions in this part. Each correct answer will receive 2 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

11. C is a point on $\overline{AD}$ and B is a point that is not on $\overleftrightarrow{AD}$. If $m\angle CAB = 65$, $m\angle CBD = 20$, and $m\angle BCD = 135$, which is the longest side of $\triangle ABC$?

12. If P is a point on the perpendicular bisector of $\overline{AB}$, prove that $\triangle ABP$ is isosceles.

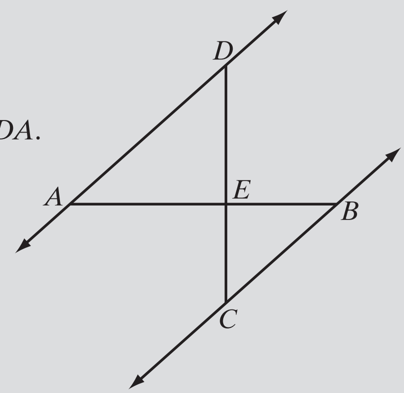
Part III

Answer all questions in this part. Each correct answer will receive 4 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

13. $ABCD$ is an equilateral quadrilateral. Prove that the diagonal, $\overline{AC}$, bisects $\angle DAB$ and $\angle DCB$.

14. $\overleftrightarrow{AEB}$ and $\overleftrightarrow{CED}$ intersect at E and $\overleftrightarrow{AD} \parallel \overleftrightarrow{CB}$.

Prove that $m\angle DEB = m\angle EBC + m\angle EDA$.



Part IV

Answer all questions in this part. Each correct answer will receive 6 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

15. The measures of the angles of a triangle are in the ratio $3:4:8$. Find the measure of the smallest exterior angle.

16. Write an equation of the perpendicular bisector of $\overline{AB}$ if the coordinates of the endpoints of $\overline{AB}$ are $A(-1, -2)$ and $B(7, 6)$.