CHAPTER

10

QUADRILATERALS

Euclid's fifth postulate was often considered to be a "flaw" in his development of geometry. Girolamo Saccheri (1667–1733) was convinced that by the application of rigorous logical reasoning, this postulate could be proved. He proceeded to develop a geometry based on an isosceles quadrilateral with two base angles that are right angles. This isosceles quadrilateral had been proposed by the Persian mathematician Nasir al-Din al-Tusi (1201–1274). Using this quadrilateral, Saccheri attempted to prove Euclid's fifth postulate by reasoning to a contradiction. After his death, his work was published under the title *Euclid Freed of Every Flaw*. Saccheri did not, as he set out to do, prove the parallel postulate, but his work laid the foundations for new geometries. János Bolyai (1802–1860) and Nicolai Lobachevsky (1793–1856) developed a geometry that allowed two lines parallel to the given line, through a point not on a given line. Georg Riemann (1826–1866) developed a geometry in which there is no line parallel to a given line through a point not on the given line.

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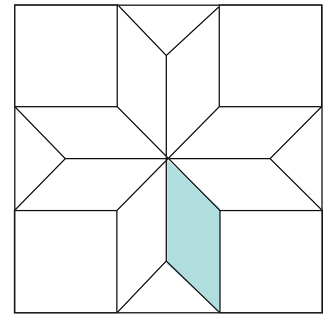
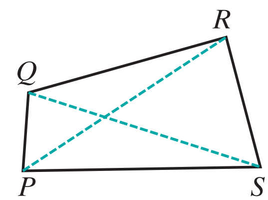
10-1 THE GENERAL QUADRILATERAL

Patchwork is an authentic American craft, developed by our frugal ancestors in a time when nothing was wasted and useful parts of discarded clothing were stitched into warm and decorative quilts.



Quilt patterns, many of which acquired names as they were handed down from one generation to the next, were the product of creative and industrious people. In the Lone Star pattern, *quadrilaterals* are arranged to form larger *quadrilaterals* that form a star. The creators of this pattern were perhaps more aware of the pleasing effect of the design than of the mathematical relationships that were the basis of the pattern.



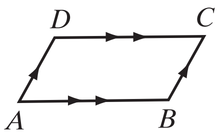
A **quadrilateral** is a polygon with four sides. In this chapter we will study the various special quadrilaterals and the properties of each. Let us first name the general parts and state properties of any quadrilateral, using $PQRS$ as an example.

- **Consecutive vertices** or **adjacent vertices** are vertices that are endpoints of the same side such as P and Q, Q and R, R and S, S and P.
- **Consecutive sides** or **adjacent sides** are sides that have a common endpoint, such as $\overline{PQ}$ and $\overline{QR}$, $\overline{QR}$ and $\overline{RS}$, $\overline{RS}$ and $\overline{SP}$, $\overline{SP}$ and $\overline{PQ}$.
- **Opposite sides of a quadrilateral** are sides that do not have a common endpoint, such as $\overline{PQ}$ and $\overline{RS}$, $\overline{SP}$ and $\overline{QR}$.
- **Consecutive angles of a quadrilateral** are angles whose vertices are consecutive, such as $\angle P$ and $\angle Q$, $\angle Q$ and $\angle R$, $\angle R$ and $\angle S$, $\angle S$ and $\angle P$.
- **Opposite angles of a quadrilateral** are angles whose vertices are not consecutive, such as $\angle P$ and $\angle R$, $\angle Q$ and $\angle S$.
- A **diagonal of a quadrilateral** is a line segment whose endpoints are two nonadjacent vertices of the quadrilateral, such as $\overline{PR}$ and $\overline{QS}$.
- The sum of the measures of the angles of a quadrilateral is 360 degrees. Therefore, $m\angle P + m\angle Q + m\angle R + m\angle S = 360$.

10-2 THE PARALLELOGRAM

DEFINITION

A **parallelogram** is a quadrilateral in which two pairs of opposite sides are parallel.

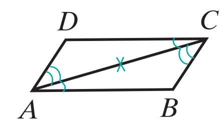


Quadrilateral $ABCD$ is a parallelogram because $\overline{AB} \parallel \overline{CD}$ and $\overline{BC} \parallel \overline{DA}$. The symbol for parallelogram $ABCD$ is $\square ABCD$.

Note the use of arrowheads, pointing in the same direction, to show sides that are parallel in the figure.

Theorem 10.1 A diagonal divides a parallelogram into two congruent triangles.

Given Parallelogram $ABCD$ with diagonal $\overline{AC}$



Prove $\triangle ABC \cong \triangle CDA$

Proof Since opposite sides of a parallelogram are parallel, alternate interior angles can be proved congruent using the diagonal as the transversal.

Statements	Reasons
1. $ABCD$ is a parallelogram.	**1.** Given.
2. $\overline{AB} \parallel \overline{CD}$ and $\overline{BC} \parallel \overline{DA}$	**2.** A parallelogram is a quadrilateral in which two pairs of opposite sides are parallel.
3. $\angle BAC \cong \angle DCA$ and $\angle BCA \cong \angle DAC$	**3.** If two parallel lines are cut by a transversal, alternate interior angles are congruent.
4. $\overline{AC} \cong \overline{AC}$	**4.** Reflexive property of congruence.
5. $\triangle ABC \cong \triangle CDA$	**5.** ASA.

We have proved that the diagonal $\overline{AC}$ divides parallelogram $ABCD$ into two congruent triangles. An identical proof could be used to show that $\overline{BD}$ divides the parallelogram into two congruent triangles, $\triangle ABD \cong \triangle CDB$.

The following corollaries result from this theorem.

Corollary 10.1a Opposite sides of a parallelogram are congruent.

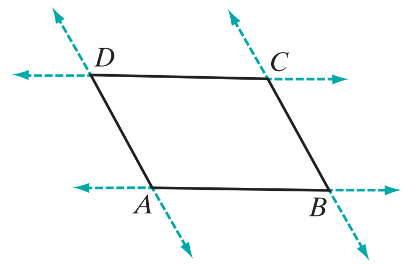
Corollary 10.1b Opposite angles of a parallelogram are congruent.

The proofs of these corollaries are left to the student. (See exercises 14 and 15.)

We can think of each side of a parallelogram as a segment of a transversal that intersects a pair of parallel lines. This enables us to prove the following theorem.

Theorem 10.2 Two consecutive angles of a parallelogram are supplementary.

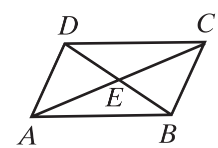
Proof In $\square ABCD$, opposite sides are parallel. If two parallel lines are cut by a transversal, then two interior angles on the same side of the transversal are supplementary. Therefore, $\angle A$ is supplementary to $\angle B$, $\angle B$ is supplementary to $\angle C$, $\angle C$ is supplementary to $\angle D$, and $\angle D$ is supplementary to $\angle A$.



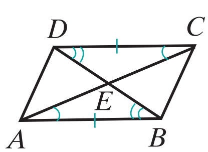
Theorem 10.3 The diagonals of a parallelogram bisect each other.

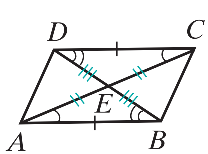
Given $\square ABCD$ with diagonals $\overline{AC}$ and $\overline{BD}$ intersecting at E.

Prove $\overline{AC}$ and $\overline{BD}$ bisect each other.



Proof





Statements	Reasons
1. $\overline{AB} \parallel \overline{CD}$	**1.** Opposite sides of a parallelogram are parallel.
2. $\angle BAE \cong \angle DCE$ and $\angle ABE \cong \angle CDE$	**2.** If two parallel lines are cut by a transversal, the alternate interior angles are congruent.
3. $\overline{AB} \cong \overline{CD}$	**3.** Opposite sides of a parallelogram are congruent.
4. $\triangle ABE \cong \triangle CDE$	**4.** ASA.
5. $\overline{AE} \cong \overline{CE}$ and $\overline{BE} \cong \overline{DE}$	**5.** Corresponding part of congruent triangles are congruent.
6. E is the midpoint of $\overline{AC}$ and of $\overline{BD}$.	**6.** The midpoint of a line segment divides the segment into two congruent segments.
7. $\overline{AC}$ and $\overline{BD}$ bisect each other.	**7.** The bisector of a line segment intersects the segment at its midpoint.

DEFINITION

The **distance between two parallel lines** is the length of the perpendicular from any point on one line to the other line.

Properties of a Parallelogram

1. Opposite sides are parallel.
2. A diagonal divides a parallelogram into two congruent triangles.
3. Opposite sides are congruent.
4. Opposite angles are congruent.
5. Consecutive angles are supplementary.
6. The diagonals bisect each other.

EXAMPLE 1

In $\square ABCD$, $m\angle B$ exceeds $m\angle A$ by 46 degrees. Find $m\angle B$.

Solution Let $x = m\angle A$.

Then $x + 46 = m\angle B$.

Two consecutive angles of a parallelogram are supplementary. Therefore,

$$\begin{aligned} m\angle A + m\angle B &= 180 \\ x + x + 46 &= 180 \\ 2x + 46 &= 180 \\ 2x &= 134 \\ x &= 67 \end{aligned} \qquad \begin{aligned} m\angle B &= x + 46 \\ &= 67 + 46 \\ &= 113 \end{aligned}$$

Answer $m\angle B = 113$

Exercises

Writing About Mathematics

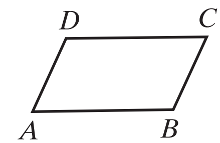
1. Theorem 10.2 states that two consecutive angles of a parallelogram are supplementary. If two opposite angles of a quadrilateral are supplementary, is the quadrilateral a parallelogram? Justify your answer.
2. A diagonal divides a parallelogram into two congruent triangles. Do two diagonals divide a parallelogram into four congruent triangles? Justify your answer.

Developing Skills

3. Find the degree measures of the other three angles of a parallelogram if one angle measures:

a. 70 **b.** 65 **c.** 90 **d.** 130 **e.** 155 **f.** 168

In 4–11, $ABCD$ is a parallelogram.



4. The degree measure of $\angle A$ is represented by $2x - 20$ and the degree measure of $\angle B$ by $2x$. Find the value of x, of m$\angle A$, and of m$\angle B$.

5. The degree measure of $\angle A$ is represented by $2x + 10$ and the degree measure of $\angle B$ by $3x$. Find the value of x, of m$\angle A$, and of m$\angle B$.

6. The measure of $\angle A$ is 30 degrees less than twice the measure of $\angle B$. Find the measure of each angle of the parallelogram.

7. The measure of $\angle A$ is represented by $x + 44$ and the measure of $\angle C$ by $3x$. Find the measure of each angle of the parallelogram.

8. The measure of $\angle B$ is represented by $7x$ and m$\angle D$ by $5x + 30$. Find the measure of each angle of the parallelogram.

9. The measure of $\angle C$ is one-half the measure of $\angle B$. Find the measure of each angle of the parallelogram.

10. If $AB = 4x + 7$ and $CD = 3x + 12$, find AB and CD.

11. If $AB = 4x + y$, $BC = y + 4$, $CD = 3x + 6$, and $DA = 2x + y$, find the lengths of the sides of the parallelogram.

12. The diagonals of $\square ABCD$ intersect at E. If $AE = 5x - 3$ and $EC = 15 - x$, find AC.

13. The diagonals of $\square ABCD$ intersect at E. If $DE = 4y + 1$ and $EB = 5y - 1$, find DB.

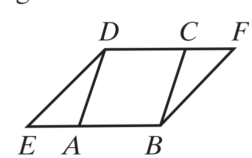
Applying Skills

14. Prove Corollary 10.1a, "Opposite sides of a parallelogram are congruent."

15. Prove Corollary 10.1b, "Opposite angles of a parallelogram are congruent."

16. *Given:* Parallelogram $EBFD$ and parallelogram $ABCD$ with $\overline{EAB}$ and $\overline{DCF}$

Prove: $\triangle EAD \cong \triangle FCB$



17. Petrina said that the floor of her bedroom is in the shape of a parallelogram and that at least one of the angles is a right angle. Show that the floor of Petrina's bedroom has four right angles.

18. The deck that Jeremiah is building is in the shape of a quadrilateral, $ABCD$. The measure of the angle at A is not equal to the measure of the angle at C. Prove that the deck is not in the shape of a parallelogram.

19. Quadrilaterals $ABCD$ and $PQRS$ are parallelograms with $\overline{AB} \cong \overline{PQ}$ and $\overline{BC} \cong \overline{QR}$. Prove that $ABCD \cong PQRS$ or draw a counterexample to show that they may not be congruent.

20. Quadrilaterals $ABCD$ and $PQRS$ are parallelograms with $\overline{AB} \cong \overline{PQ}$, $\overline{BC} \cong \overline{QR}$, and $\angle B \cong \angle Q$. Prove that $ABCD \cong PQRS$ or draw a counterexample to show that they may not be congruent.

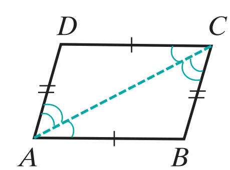
10-3 PROVING THAT A QUADRILATERAL IS A PARALLELOGRAM

If we wish to prove that a certain quadrilateral is a parallelogram, we can do so by proving its opposite sides are parallel, thus satisfying the definition of a parallelogram. Now we want to determine other ways of proving that a quadrilateral is a parallelogram.

Theorem 10.4 If both pairs of opposite sides of a quadrilateral are congruent, the quadrilateral is a parallelogram.

Given Quadrilateral $ABCD$ with $\overline{AB} \cong \overline{CD}$, $\overline{AD} \cong \overline{BC}$

Prove $ABCD$ is a parallelogram.

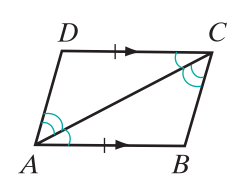


Proof In $ABCD$, $\overline{AC}$ is a diagonal. Triangles ABC and CDA are congruent by SSS. Corresponding parts of congruent triangles are congruent, so $\angle BAC \cong \angle DCA$ and $\angle DAC \cong \angle ACB$. $\overline{AC}$ is a transversal that cuts $\overline{AB}$ and $\overline{DC}$. Alternate interior angles $\angle BAC$ and $\angle DCA$ are congruent, so $\overline{AB} \parallel \overline{DC}$. $\overline{AC}$ is also a transversal that cuts $\overline{AD}$ and $\overline{BC}$. Alternate interior angles $\angle DAC$ and $\angle ACB$ are congruent so $\overline{AD} \parallel \overline{BC}$. Therefore, $ABCD$ is a parallelogram.

Theorem 10.5 If one pair of opposite sides of a quadrilateral is both congruent and parallel, the quadrilateral is a parallelogram.

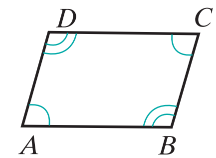
Given Quadrilateral $ABCD$ with $\overline{AB} \parallel \overline{CD}$ and $\overline{AB} \cong \overline{CD}$

Prove $ABCD$ is a parallelogram.



Proof Since $\overline{AB}$ is parallel to $\overline{CD}$, $\angle BAC$ and $\angle DCA$ are a pair of congruent alternate interior angles. Therefore, by SAS, $\triangle DCA \cong \triangle BAC$. Corresponding parts of congruent triangles are congruent, so $\angle DAC \cong \angle ACB$. $\overline{AC}$ is a transversal that cuts $\overline{AD}$ and $\overline{BC}$, forming congruent alternate interior angles $\angle DAC$ and $\angle ACB$. Therefore, $\overline{AD} \parallel \overline{BC}$, and $ABCD$ is a parallelogram.

Theorem 10.6 If both pairs of opposite angles of a quadrilateral are congruent, the quadrilateral is a parallelogram.



Given Quadrilateral $ABCD$ with $\angle A \cong \angle C$ and $\angle B \cong \angle D$

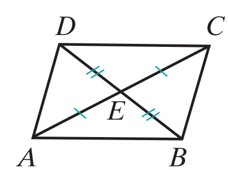
Prove $ABCD$ is a parallelogram.

Proof The sum of the measures of the angles of a quadrilateral is 360 degrees. Therefore, $m\angle A + m\angle B + m\angle C + m\angle D = 360$. It is given that $\angle A \cong \angle C$ and $\angle B \cong \angle D$. Congruent angles have equal measures so $m\angle A = m\angle C$ and $m\angle B = m\angle D$.

By substitution, $m\angle A + m\angle D + m\angle A + m\angle D = 360$. Then, $2m\angle A + 2m\angle D = 360$ or $m\angle A + m\angle D = 180$. Similarly, $m\angle A + m\angle B = 180$. If the sum of the measures of two angles is 180, the angles are supplementary. Therefore, $\angle A$ and $\angle D$ are supplementary and $\angle A$ and $\angle B$ are supplementary.

Two coplanar lines are parallel if a pair of interior angles on the same side of the transversal are supplementary. Therefore, $\overline{AB} \parallel \overline{DC}$ and $\overline{AD} \parallel \overline{BC}$. Quadrilateral $ABCD$ is a parallelogram because it has two pairs of parallel sides. ■

Theorem 10.7 If the diagonals of a quadrilateral bisect each other, the quadrilateral is a parallelogram.



Given Quadrilateral $ABCD$ with $\overline{AC}$ and $\overline{BD}$ intersecting at E, $\overline{AE} \cong \overline{EC}$, $\overline{BE} \cong \overline{ED}$

Prove $ABCD$ is a parallelogram.

Strategy Prove that $\triangle ABE \cong \triangle CDE$ to show that one pair of opposite sides is congruent and parallel.

The proof of Theorem 10.7 is left to the student. (See exercise 15.)

SUMMARY To prove that a quadrilateral is a parallelogram, prove that any one of the following statements is true:

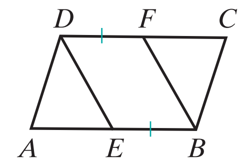
1. Both pairs of opposite sides are parallel.
2. Both pairs of opposite sides are congruent.
3. One pair of opposite sides is both congruent and parallel.
4. Both pairs of opposite angles are congruent.
5. The diagonals bisect each other.

EXAMPLE 1

Given: $ABCD$ is a parallelogram.

E is on $\overline{AB}$, F is on $\overline{DC}$, and $\overline{EB} \cong \overline{DF}$.

Prove: $\overline{DE} \parallel \overline{FB}$



Proof We will prove that $EBFD$ is a parallelogram.

Statements	Reasons
1. $ABCD$ is a parallelogram.	**1.** Given.
2. $\overline{AB} \parallel \overline{DC}$	**2.** Opposite sides of a parallelogram are parallel.
3. $\overline{EB} \parallel \overline{DF}$	**3.** Segments of parallel lines are parallel.
4. $\overline{EB} \cong \overline{DF}$	**4.** Given.
5. $EBFD$ is a parallelogram.	**5.** If one pair of opposite sides of a quadrilateral is both congruent and parallel, the quadrilateral is a parallelogram.
6. $\overline{DE} \parallel \overline{FB}$	**6.** Opposite sides of a parallelogram are parallel.

Exercises

Writing About Mathematics

1. What statement and reason can be added to the proof in Example 1 to prove that $\overline{DE} \cong \overline{FB}$?

2. What statement and reason can be added to the proof in Example 1 to prove that $\angle DEB \cong \angle BFD$?

Developing Skills

In 3–7, in each case, the *given* is marked on the figure. Tell why each quadrilateral $ABCD$ is a parallelogram.

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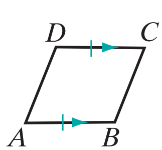

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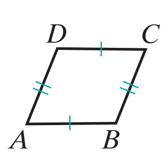

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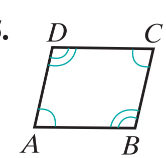

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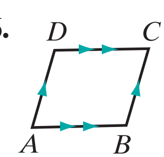

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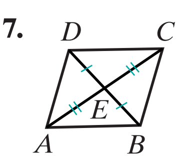


8. $ABCD$ is a quadrilateral with $\overline{AB} \parallel \overline{CD}$ and $\angle A \cong \angle C$. Prove that $ABCD$ is a parallelogram.

9. $PQRS$ is a quadrilateral with $\angle P \cong \angle R$ and $\angle P$ the supplement of $\angle Q$. Prove that $PQRS$ is a parallelogram.

10. $DEFG$ is a quadrilateral with $\overline{DF}$ drawn so that $\angle FDE \cong \angle DFG$ and $\angle GDF \cong \angle EFD$. Prove that $DEFG$ is a parallelogram.

11. $ABCD$ is a parallelogram. E is the midpoint of $\overline{AB}$ and F is the midpoint of $\overline{CD}$. Prove that $AEFD$ is a parallelogram.

12. $EFGH$ is a parallelogram and J is a point on $\overrightarrow{EF}$ such that F is the midpoint of $\overline{EJ}$. Prove that $FJGH$ is a parallelogram.

13. $ABCD$ is a parallelogram. The midpoint of $\overline{AB}$ is P, the midpoint of $\overline{BC}$ is Q, the midpoint of $\overline{CD}$ is R, and the midpoint of $\overline{DA}$ is S.

a. Prove that $\triangle APS \cong \triangle CRQ$ and that $\triangle BQP \cong \triangle DSR$.

b. Prove that $PQRS$ is a parallelogram.

14. A quadrilateral has three right angles. Is the quadrilateral a parallelogram? Justify your answer.

Applying Skills

15. Prove Theorem 10.7, "If the diagonals of a quadrilateral bisect each other, the quadrilateral is a parallelogram."

16. Prove that a parallelogram can be drawn by joining the endpoints of two line segments that bisect each other.

17. The vertices of quadrilateral $ABCD$ are $A(-2, 1)$, $B(4, -2)$, $C(8, 2)$, and $D(2, 5)$. Is $ABCD$ a parallelogram? Justify your answer.

18. Farmer Brown's pasture is in the shape of a quadrilateral, $PQRS$. The pasture is crossed by two diagonal paths, $\overline{PR}$ and $\overline{QS}$. The quadrilateral is not a parallelogram. Show that the paths do not bisect each other.

19. Toni cut two congruent scalene triangles out of cardboard. She labeled one triangle $\triangle ABC$ and the other $\triangle A'B'C'$ so that $\triangle ABC \cong \triangle A'B'C'$. She placed the two triangles next to each other so that A coincided with B' and B coincided with A'. Was the resulting quadrilateral $ACBC'$ a parallelogram? Prove your answer.

20. Quadrilateral $ABCD$ is a parallelogram with M the midpoint of $\overline{AB}$ and N the midpoint of $\overline{CD}$.

a. Prove that $AMND$ and $MBCN$ are parallelograms.

b. Prove that $AMND$ and $MBCN$ are congruent.

10-4 THE RECTANGLE

DEFINITION

A **rectangle** is a parallelogram containing one right angle.

If one angle, $\angle A$, of a parallelogram $ABCD$ is a right angle, then $\square ABCD$ is a rectangle.

Any side of a rectangle may be called the *base* of the rectangle. Thus, if side $\overline{AB}$ is taken as the base, then either consecutive side, $\overline{AD}$ or $\overline{BC}$, is called the *altitude* of the rectangle.

Since a rectangle is a special kind of parallelogram, a rectangle has all the properties of a parallelogram. In addition, we can prove two special properties for the rectangle.

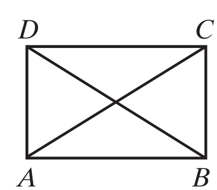
Theorem 10.8 All angles of a rectangle are right angles.

Given Rectangle $ABCD$ with $\angle A$ a right angle.

Prove $\angle B$, $\angle C$, and $\angle D$ are right angles.

Proof By definition, rectangle $ABCD$ is a parallelogram. Opposite angles of a parallelogram are congruent, so $\angle A \cong \angle C$. Angle A is a right angle so $\angle C$ is a right angle. Consecutive angles of a parallelogram are supplementary. Therefore, since $\angle A$ and $\angle B$ are supplementary and $\angle A$ is right angle, $\angle B$ is also a right angle. Similarly, since $\angle C$ and $\angle D$ are supplementary and $\angle C$ is a right angle, $\angle D$ is also a right angle.

Theorem 10.9 The diagonals of a rectangle are congruent.



Given $ABCD$ is a rectangle.

Prove $\overline{AC} \cong \overline{BD}$

Strategy Prove $\triangle DAB \cong \triangle CBA$.

The proof of Theorem 10.9 is left to the student. (See exercise 12.)

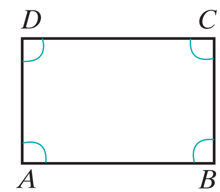
Properties of a Rectangle

1. A rectangle has all the properties of a parallelogram.
2. A rectangle has four right angles and is therefore equiangular.
3. The diagonals of a rectangle are congruent.

Proving That a Quadrilateral Is a Rectangle

We prove that a quadrilateral is a rectangle by showing that it has the special properties of a rectangle. For example:

Theorem 10.10 If a quadrilateral is equiangular, then it is a rectangle.

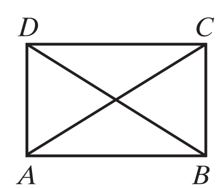


Given Quadrilateral $ABCD$ with $\angle A \cong \angle B \cong \angle C \cong \angle D$.

Prove $ABCD$ is a rectangle.

Proof Quadrilateral $ABCD$ is a parallelogram because the opposite angles are congruent. The sum of the measures of the angles of a quadrilateral is 360 degrees. Thus, the measure of each of the four congruent angles is 90 degrees. Therefore, $ABCD$ is a rectangle because it is a parallelogram with a right angle.

Theorem 10.11 If the diagonals of a parallelogram are congruent, the parallelogram is a rectangle.



Given Parallelogram $ABCD$ with $\overline{AC} \cong \overline{BD}$

Prove $ABCD$ is a rectangle.

Strategy Prove that $\triangle DAB \cong \triangle CBA$. Therefore, $\angle DAB$ and $\angle CBA$ are both congruent and supplementary.

The proof of Theorem 10.11 is left to the student. (See exercise 13.)

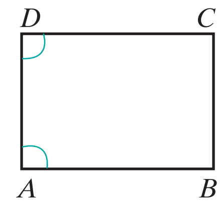
SUMMARY To prove that a quadrilateral is a rectangle, prove that any one of the following statements is true:

1. The quadrilateral is a parallelogram with one right angle.
2. The quadrilateral is equiangular.
3. The quadrilateral is a parallelogram whose diagonals are congruent.

EXAMPLE 1

Given: $ABCD$ is a parallelogram with $m\angle A = m\angle D$.

Prove: $ABCD$ is a rectangle.



Proof We will prove that $ABCD$ has a right angle.

Statements	Reasons
1. $ABCD$ is a parallelogram.	**1.** Given.
2. $\overline{AB} \parallel \overline{CD}$	**2.** A parallelogram is a quadrilateral in which two pairs of opposite sides are parallel.
3. $\angle A$ and $\angle D$ are supplementary.	**3.** Two consecutive angles of a parallelogram are supplementary.
4. $m\angle A + m\angle D = 180$	**4.** Supplementary angles are two angles the sum of whose measures is 180.
5. $m\angle A = m\angle D$	**5.** Given.
6. $m\angle A + m\angle A = 180$ or $2m\angle A = 180$	**6.** A quantity may be substituted for its equal.
7. $m\angle A = 90$	**7.** Division postulate.
8. $ABCD$ is a rectangle.	**8.** A rectangle is a parallelogram with a right angle.

EXAMPLE 2

The lengths of diagonals of a rectangle are represented by $7x$ centimeters and $5x + 12$ centimeters. Find the length of each diagonal.

Solution The diagonals of a rectangle are congruent and therefore equal in length.

$$7x = 5x + 12$$
$$2x = 12$$
$$x = 6$$

$$7x = 7(6) \qquad\qquad 5x + 12 = 5(6) + 12$$
$$= 42 \qquad\qquad = 30 + 12$$
$$= 42$$

Answer The length of each diagonal is 42 centimeters.

Exercises

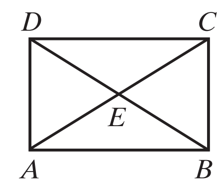
Writing About Mathematics

1. Pauli said that if one angle of a parallelogram is not a right angle, then the parallelogram is not a rectangle. Do you agree with Pauli? Explain why or why not.

2. Cindy said that if two congruent line segments intersect at their midpoints, then the quadrilateral formed by joining the endpoints of the line segments in order is a rectangle. Do you agree with Cindy? Explain why or why not.

Developing Skills

In 3–10, the diagonals of rectangle $ABCD$ intersect at E.



3. Prove that $\triangle ABE$ is isosceles.

4. $AC = 4x + 6$ and $BD = 5x - 2$. Find AC, BD, AE, and BE.

5. $AE = y + 12$ and $DE = 3y - 8$. Find AE, DE, AC, and BD.

6. $BE = 3a + 1$ and $ED = 6a - 11$. Find BE, ED, BD, and AC.

7. $AE = x + 5$ and $BD = 3x - 2$. Find AE, BD, AC, and BE.

8. $m\angle CAB = 35$. Find $m\angle CAD$, $m\angle ACB$, $m\angle AEB$, and $\angle AED$.

9. $m\angle AEB = 3x$ and $m\angle DEC = x + 80$. Find $m\angle AEB$, $m\angle DEC$, $m\angle CAB$, and $m\angle CAD$.

10. $m\angle AED = y + 10$ and $m\angle AEB = 4y - 30$. Find $m\angle AED$, $m\angle AEB$, $m\angle BAC$, and $m\angle CAD$.

Applying Skills

11. Write a coordinate proof of Theorem 10.8, "All angles of a rectangle are right angles." Let the vertices of the rectangle be $A(0, 0)$, $B(b, 0)$, $C(b, c)$, and $D(0, c)$.

12. Prove Theorem 10.9, "The diagonals of a rectangle are congruent."

13. Prove Theorem 10.11, "If the diagonals of a parallelogram are congruent, the parallelogram is a rectangle."

14. If $PQRS$ is a rectangle and M is the midpoint of $\overline{RS}$, prove that $\overline{PM} \cong \overline{QM}$.

15. The coordinates of the vertices of $ABCD$ are $A(-2, 0)$, $B(2, -2)$, $C(5, 4)$, and $D(1, 6)$.

a. Prove that $ABCD$ is a rectangle.

b. What are the coordinates of the point of intersection of the diagonals?

c. The vertices of $A'B'C'D'$ are $A'(0, -2)$, $B'(2, 2)$, $C'(-4, 5)$, and $D'(-6, 1)$. Under what specific transformation is $A'B'C'D'$ the image of $ABCD$?

d. Prove that $A'B'C'D'$ is a rectangle.

16. The coordinates of the vertices of $PQRS$ are $P(-2, 1)$, $Q(1, -3)$, $R(5, 1)$, and $S(2, 5)$.

a. Prove that $PQRS$ is a parallelogram.

b. Prove that $PQRS$ is not a rectangle.

c. $P'Q'R'S'$ is the image of $PQRS$ under $T_{-3,-3} \circ r_{y=x}$. What are the coordinates of $P'Q'R'S'$?

d. Prove that $P'Q'R'S'$ is congruent to $PQRS$.

17. Angle A in quadrilateral $ABCD$ is a right angle and quadrilateral $ABCD$ is not a rectangle. Prove that $ABCD$ is not a parallelogram.

18. Tracy wants to build a rectangular pen for her dog. She has a tape measure, which enables her to make accurate measurements of distance, but has no way of measuring angles. She places two stakes in the ground to represent opposite corners of the pen. How can she find two points at which to place stakes for the other two corners?

19. Archie has a piece of cardboard from which he wants to cut a rectangle with a diagonal that measures 12 inches. On one edge of the cardboard Archie marks two points that are less than 12 inches apart to be the endpoints of one side of the rectangle. Explain how Archie can use two pieces of string that are each 12 inches long to find the other two vertices of the rectangle.

10-5 THE RHOMBUS

DEFINITION

A **rhombus** is a parallelogram that has two congruent consecutive sides.

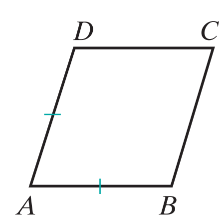
If the consecutive sides $\overline{AB}$ and $\overline{AD}$ of parallelogram $ABCD$ are congruent (that is, if $\overline{AB} \cong \overline{AD}$), then $\square ABCD$ is a rhombus.

Since a rhombus is a special kind of parallelogram, a rhombus has all the properties of a parallelogram. In addition, we can prove three special properties for the rhombus.

Theorem 10.12 All sides of a rhombus are congruent.

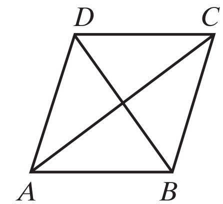
Given $ABCD$ is a rhombus with $\overline{AB} \cong \overline{DA}$.

Prove $\overline{AB} \cong \overline{BC} \cong \overline{CD} \cong \overline{DA}$



Proof By definition, rhombus $ABCD$ is a parallelogram. It is given that $\overline{AB} \cong \overline{DA}$. Opposite sides of a parallelogram are congruent, so $\overline{AB} \cong \overline{CD}$ and $\overline{BC} \cong \overline{DA}$. Using the transitive property of congruence, $\overline{AB} \cong \overline{BC} \cong \overline{CD} \cong \overline{DA}$.

Theorem 10.13 The diagonals of a rhombus are perpendicular to each other.

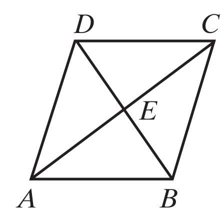


Given Rhombus $ABCD$

Prove $\overline{AC} \perp \overline{BD}$

Proof By Theorem 10.12, all sides of a rhombus are congruent. Segments that are congruent are equal. Consider the diagonal $\overline{AC}$. Point B is equidistant from the endpoints A and C since $BA = BC$. Point D is also equidistant from the endpoints A and C since $DA = DC$. If two points are each equidistant from the endpoints of a line segment, the points determine the perpendicular bisector of the line segment. Therefore, $\overline{AC} \perp \overline{BD}$. ∎

Theorem 10.14 The diagonals of a rhombus bisect its angles.



Given Rhombus $ABCD$

Prove $\overline{AC}$ bisects $\angle DAB$ and $\angle DCB$ and $\overline{DB}$ bisects $\angle CDA$ and $\angle CBA$.

Strategy Show that the diagonals separate the rhombus into four congruent triangles.

The proof of this theorem is left to the student. (See exercise 16.)

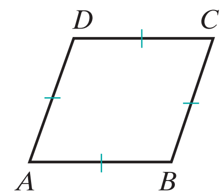
Properties of a Rhombus

1. A rhombus has all the properties of a parallelogram.
2. A rhombus is equilateral.
3. The diagonals of a rhombus are perpendicular to each other.
4. The diagonals of a rhombus bisect its angles.

Methods of Proving That a Quadrilateral Is a Rhombus

We prove that a quadrilateral is a rhombus by showing that it has the special properties of a rhombus.

Theorem 10.15 If a quadrilateral is equilateral, then it is a rhombus.

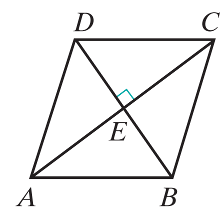


Given Quadrilateral $ABCD$ with $\overline{AB} \cong \overline{BC} \cong \overline{CD} \cong \overline{DA}$

Prove $ABCD$ is a rhombus.

Proof It is given that in $ABCD$, $\overline{AB} \cong \overline{BC} \cong \overline{CD} \cong \overline{DA}$. Since both pairs of opposite sides are congruent, $ABCD$ is a parallelogram. Two consecutive sides of parallelogram $ABCD$ are congruent, so by definition, $ABCD$ is a rhombus. ◻

Theorem 10.16 If the diagonals of a parallelogram are perpendicular to each other, the parallelogram is a rhombus.



Given Parallelogram $ABCD$ with $\overline{AC} \perp \overline{BD}$

Prove $ABCD$ is a rhombus.

Strategy The diagonals divide parallelogram $ABCD$ into four triangles. Prove that two of the triangles that share a common side are congruent. Then use the fact that corresponding parts of congruent triangles are congruent to show that parallelogram $ABCD$ has two congruent consecutive sides.

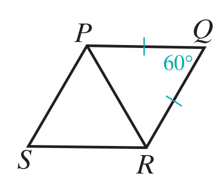
The proof of this theorem is left to the student. (See exercise 17.)

SUMMARY To prove that a quadrilateral is a rhombus, prove that any one of the following statements is true:

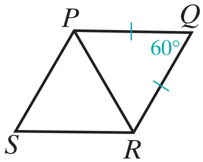
1. The quadrilateral is a parallelogram with two congruent consecutive sides.
2. The quadrilateral is equilateral.
3. The quadrilateral is a parallelogram whose diagonals are perpendicular to each other.

EXAMPLE 1

$PQRS$ is a rhombus and $m\angle PQR = 60$. Prove that $\overline{PR}$ divides the rhombus into two equilateral triangles.



Proof Since all sides of a rhombus are congruent, we know that $\overline{PQ} \cong \overline{RQ}$. Thus, $\triangle PQR$ is isosceles and its base angles are equal in measure.



Let $m\angle PRQ = m\angle RPQ = x$.

$$\begin{aligned} x + x + 60 &= 180 \\ 2x + 60 &= 180 \\ 2x &= 120 \\ x &= 60 \end{aligned}$$

Therefore, $m\angle PRQ = 60$, $m\angle RPQ = 60$, and $m\angle PQR = 60$. Triangle PQR is equilateral since an equiangular triangle is equilateral.

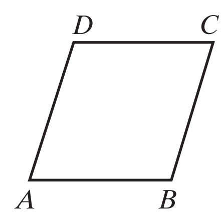
Since opposite angles of a rhombus have equal measures, $m\angle RSP = 60$. By similar reasoning, $\triangle RSP$ is equilateral.

EXAMPLE 2

Given: $ABCD$ is a parallelogram.

$AB = 2x + 1$, $DC = 3x - 11$, $AD = x + 13$.

Prove: $ABCD$ is a rhombus.



Proof (1) Since $ABCD$ is a parallelogram, opposite sides are equal in length:

$$\begin{aligned} DC &= AB \\ 3x - 11 &= 2x + 1 \\ x &= 12 \end{aligned}$$

(2) Substitute $x = 12$ to find the lengths of sides $\overline{AB}$ and $\overline{AD}$:

$$\begin{aligned} AB &= 2x + 1 \\ &= 2(12) + 1 \\ &= 25 \end{aligned} \qquad \begin{aligned} AD &= x + 13 \\ &= 12 + 13 \\ &= 25 \end{aligned}$$

(3) Since $ABCD$ is a parallelogram with two congruent consecutive sides, $ABCD$ is a rhombus.

Exercises

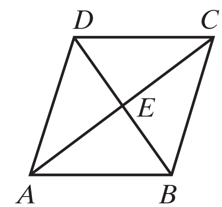
Writing About Mathematics

1. Rochelle said that the diagonals of a rhombus separate the rhombus into four congruent right triangles. Do you agree with Rochelle? Explain why or why not.

2. Concepta said that if the lengths of the diagonals of a rhombus are represented by d_1 and d_2, then a formula for the area of a rhombus is $A = \frac{1}{2}d_1d_2$. Do you agree with Concepta? Explain why or why not.

Developing Skills

In 3–10, the diagonals of rhombus $ABCD$ intersect at E.



3. Name four congruent line segments.

4. Name two pairs of congruent line segments.

5. Name a pair of perpendicular line segments.

6. Name four right angles.

7. Under a rotation of 90° about E, does A map to B? does B map to C? Justify your answer.

8. Does rhombus $ABCD$ have rotational symmetry under a rotation of 90° about E?

9. Under a reflection in E, name the image of A, of B, of C, of D, and of E.

10. Does rhombus $ABCD$ have point symmetry under a reflection in E?

11. In rhombus $PQRS$, m$\angle P$ is 120.

a. Prove that the diagonal $\overline{PR}$ separates the rhombus into two equilateral triangles.

b. If $PR = 24$ cm, what is the length of each side of the rhombus?

In 12–15, tell whether each conclusion follows from the given premises. If not, draw a counterexample.

12. In a parallelogram, opposite sides are congruent.
A rhombus is a parallelogram.
Conclusion: In a rhombus, opposite sides are congruent.

13. In a rhombus, diagonals are perpendicular to each other.
A rhombus is a parallelogram.
Conclusion: In a parallelogram, diagonals are perpendicular to each other.

14. The diagonals of a rhombus bisect the angles of the rhombus.
A rhombus is a parallelogram.
Conclusion: The diagonals of a parallelogram bisect its angles.

15. Consecutive angles of a parallelogram are supplementary.
A rhombus is a parallelogram.
Conclusion: Consecutive angles of a rhombus are supplementary.

Applying Skills

16. Prove Theorem 10.14, "The diagonals of a rhombus bisect its angles."

17. Prove Theorem 10.16, "If the diagonals of a parallelogram are perpendicular to each other, the parallelogram is a rhombus."

18. The vertices of quadrilateral $ABCD$ are $A(-1, -1)$, $B(4, 0)$, $C(5, 5)$, and $D(0, 4)$.

a. Prove that $ABCD$ is a parallelogram.

b. Prove that the diagonals of $ABCD$ are perpendicular.

c. Is $ABCD$ a rhombus? Justify your answer.

19. If a diagonal separates a quadrilateral $KLMN$ into two equilateral triangles, prove that $KLMN$ is a rhombus with the measure of one angle equal to 60 degrees.

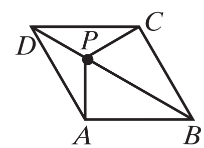
20. Prove that the diagonals of a rhombus separate the rhombus into four congruent right triangles.

21. Prove that if the diagonals of a quadrilateral are the perpendicular bisectors of each other, the quadrilateral is a rhombus.

22. Prove that if the diagonals of a parallelogram bisect the angles of the parallelogram, then the parallelogram is a rhombus.

23. Let P be any point on diagonal $\overline{BD}$ of rhombus $ABCD$.

Prove that $\overline{AP} \cong \overline{CP}$.



24. The vertices of quadrilateral $ABCD$ are $A(-2, -1)$, $B(2, -4)$, $C(2, 1)$, $D(-2, 4)$.

a. Prove that $ABCD$ is a parallelogram.

b. Find the coordinates of E, the point of intersection of the diagonals.

c. Prove that the diagonals are perpendicular.

d. Is $ABCD$ a rhombus? Justify your answer.

25. $ABCD$ is a parallelogram. The midpoint of $\overline{AB}$ is M, the midpoint of $\overline{CD}$ is N, and $AM = AD$.

a. Prove that $AMND$ is a rhombus.

b. Prove that $MBCN$ is a rhombus.

c. Prove that $AMND$ is congruent to $MBCN$.

Hands-On Activity

In this activity, you will construct a rhombus given the diagonal. You may use geometry software or compass and straightedge.

1. First draw a segment measuring 12 centimeters. This will be the diagonal of the rhombus. The endpoints of this segment are opposite vertices of the rhombus.
2. Now construct the perpendicular bisector of this segment.
3. Show that you can choose any point on that perpendicular bisector as a third vertex of the rhombus.
4. How can you determine the fourth vertex of the rhombus?
5. Compare the rhombus you constructed with rhombuses constructed by your classmates. How are they alike? How are they different?

10-6 THE SQUARE

DEFINITION

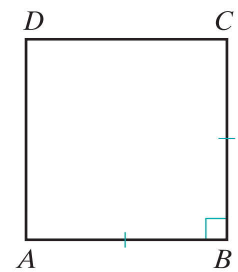
A **square** is a rectangle that has two congruent consecutive sides.

If consecutive sides $\overline{AB}$ and $\overline{AD}$ of rectangle $ABCD$ are congruent (that is, if $\overline{AB} \cong \overline{AD}$), then rectangle $ABCD$ is a square.

Theorem 10.17 A square is an equilateral quadrilateral.

Given $ABCD$ is a square with $\overline{AB} \cong \overline{BC}$.

Prove $\overline{AB} \cong \overline{BC} \cong \overline{CD} \cong \overline{DA}$



Proof A square is a rectangle and a rectangle is a parallelogram, so $ABCD$ is a parallelogram. It is given that $\overline{AB} \cong \overline{BC}$. Opposite sides of a parallelogram are congruent, so $\overline{AB} \cong \overline{CD}$ and $\overline{BC} \cong \overline{DA}$. Using the transitive property of congruence, $\overline{AB} \cong \overline{BC} \cong \overline{CD} \cong \overline{DA}$.

Theorem 10.18 A square is a rhombus.

Given Square $ABCD$

Prove $ABCD$ is a rhombus.

Proof A square is an equilateral quadrilateral. If a quadrilateral is equilateral, then it is a rhombus. Therefore, $ABCD$ is a rhombus.

Properties of a Square

1. A square has all the properties of a rectangle.
2. A square has all the properties of a rhombus.

Methods of Proving That a Quadrilateral Is a Square

We prove that a quadrilateral is a square by showing that it has the special properties of a square.

Theorem 10.19 If one of the angles of a rhombus is a right angle, then the rhombus is a square.

Given $ABCD$ is a rhombus with $\angle A$ a right angle.

Prove $ABCD$ is a square.

Strategy Show that $ABCD$ is a rectangle and that $\overline{AB} \cong \overline{BC}$.

The proof of this theorem is left to the student. (See exercise 12.)

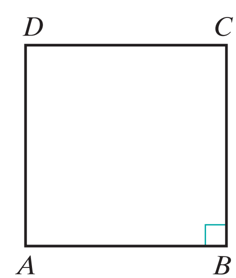
SUMMARY
To prove that a quadrilateral is a square, prove either of the following statements:

1. The quadrilateral is a rectangle in which two consecutive sides are congruent.
2. The quadrilateral is a rhombus one of whose angles is a right angle.

EXAMPLE 1

Given: Quadrilateral $ABCD$ is equilateral and $\angle ABC$ is a right angle.

Prove: $ABCD$ is a square.



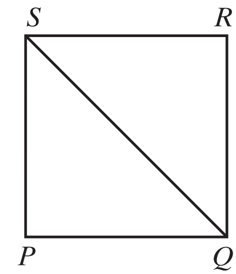
Proof

Statements	Reasons
1. $ABCD$ is equilateral.	**1.** Given
2. $ABCD$ is a rhombus.	**2.** If a quadrilateral is equilateral, then it is a rhombus.
3. $\angle ABC$ is a right angle.	**3.** Given.
4. $ABCD$ is a square.	**4.** If one of the angles of a rhombus is a right angle, then the rhombus is a square.

EXAMPLE 2

In square $PQRS$, $\overline{SQ}$ is a diagonal.

Find m$\angle PSQ$.



Solution A square is a rhombus, and the diagonal of a rhombus bisects its angles. Therefore, the diagonal $\overline{SQ}$ bisects $\angle PSR$. Since a square is a rectangle, $\angle PSR$ is a right angle and m$\angle PSR = 90$. Therefore, m$\angle PSQ = \frac{1}{2}(90) = 45$.

Exercises

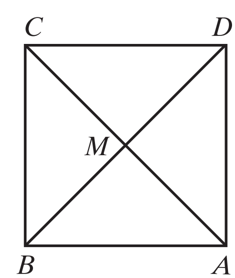
Writing About Mathematics

1. Ava said that the diagonals of a square separate the square into four congruent isosceles right triangles. Do you agree with Ava? Justify your answer.

2. Raphael said that a square could be defined as a quadrilateral that is both equiangular and equilateral. Do you agree with Raphael? Justify your answer.

Developing Skills

In 3–6, the diagonals of square $ABCD$ intersect at M.



3. Prove that $\overline{AC}$ is the perpendicular bisector of $\overline{BD}$.

4. If $AC = 3x + 2$ and $BD = 7x - 10$, find AC, BD, AM, BM.

5. If $AB = a + b$, $BC = 2a$, and $CD = 3b - 5$, find AB, BC, CD, and DA.

6. If m$\angle AMD = x + 2y$ and m$\angle ABC = 2x - y$, find the values of x and y.

In 7–11, tell whether each conclusion follows from the given premises. If not, draw a counterexample.

7. If a quadrilateral is a square, then all sides are congruent.
If all sides of a quadrilateral are congruent then it is a rhombus.
Conclusion: If a quadrilateral is a square, then it is a rhombus.

8. A diagonal of a parallelogram separates the parallelogram into two congruent triangles.
A square is a parallelogram.
Conclusion: A diagonal of a square separates the square into two congruent triangles.

9. If a quadrilateral is a square, then all angles are right angles.
If a quadrilateral is a square, then it is a rhombus.
Conclusion: In a rhombus, all angles are right angles.

10. If a quadrilateral is a square, then it is a rectangle.
If a quadrilateral is a rectangle, then it is a parallelogram.
Conclusion: If a quadrilateral is a square, then it is a parallelogram.

11. If a quadrilateral is a square, then it is equilateral.
If a quadrilateral is a square, then it is a rectangle.
Conclusion: If a quadrilateral is equilateral, then it is a rectangle.

Applying Skills

12. Prove Theorem 10.19, "If one of the angles of a rhombus is a right angle, then the rhombus is a square."

13. Prove that the diagonals of a square are perpendicular to each other.

14. Prove that the diagonals of a square divide the square into four congruent isosceles right triangles.

15. Two line segments, $\overline{AEC}$ and $\overline{BED}$, are congruent. Each is the perpendicular bisector of the other. Prove that $ABCD$ is a square.

16. Prove that if the midpoints of the sides of a square are joined in order, another square is formed.

17. The vertices of quadrilateral $PQRS$ are $P(1, 1)$, $Q(4, -2)$, $R(7, 1)$, and $S(4, 4)$.

a. Prove that the diagonals of the quadrilateral bisect each other.

b. Prove that the diagonals of the quadrilateral are perpendicular to each other.

c. Is the quadrilateral a square? Justify your answer.

d. The vertices of $P'Q'R'S'$ are $P'(-1, 1)$, $Q'(2, 4)$, $R'(-1, 7)$, and $S'(-4, 4)$. Under what specific transformation is $P'Q'R'S'$ the image of $PQRS$?

18. The vertices of quadrilateral $ABCD$ are $A(-3, 2)$, $B(1, -2)$, $C(5, 2)$, and $D(1, 6)$.

a. Prove that the diagonals of the quadrilateral bisect each other.

b. Prove that the diagonals of the quadrilateral are perpendicular to each other.

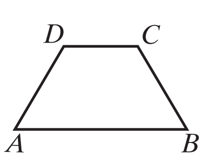
c. Is the quadrilateral a square? Justify your answer.

d. The vertices of $A'B'C'D'$ are $A'(-6, 3)$, $B'(-2, -1)$, $C'(2, 3)$, and $D'(-2, 7)$. Under what specific transformation is $A'B'C'D'$ the image of $ABCD$?

10-7 THE TRAPEZOID

DEFINITION

A **trapezoid** is a quadrilateral in which two and only two sides are parallel.

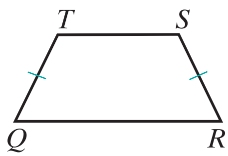


If $\overline{AB} \parallel \overline{DC}$ and $\overline{AD}$ is not parallel to $\overline{BC}$, then quadrilateral $ABCD$ is a trapezoid. The parallel sides, $\overline{AB}$ and $\overline{DC}$, are called the **bases** of the trapezoid; the nonparallel sides, $\overline{AD}$ and $\overline{BC}$, are called the **legs** of the trapezoid.

The Isosceles Trapezoid and Its Properties

DEFINITION

An **isosceles trapezoid** is a trapezoid in which the nonparallel sides are congruent.



If $\overline{TS} \parallel \overline{QR}$ and $\overline{QT} \cong \overline{RS}$, then $QRST$ is an isosceles trapezoid. The angles whose vertices are the endpoints of a base are called **base angles**. Here, $\angle Q$ and $\angle R$ are one pair of base angles because Q and R are endpoints of base $\overline{QR}$. Also, $\angle T$ and $\angle S$ are a second pair of base angles because T and S are endpoints of base $\overline{TS}$.

Proving That a Quadrilateral Is an Isosceles Trapezoid

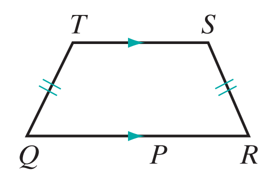
We prove that a quadrilateral is an isosceles trapezoid by showing that it satisfies the conditions of the definition of an isosceles trapezoid: only two sides are parallel and the nonparallel sides are congruent.

We may also prove special theorems for an isosceles trapezoid.

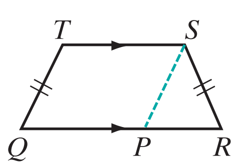
Theorem 10.20a If a trapezoid is isosceles, then the base angles are congruent.

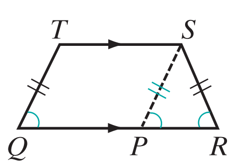
Given Isosceles trapezoid $QRST$ with $\overline{QR} \parallel \overline{ST}$ and $\overline{TQ} \cong \overline{SR}$

Prove $\angle Q \cong \angle R$



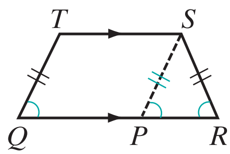
Proof





Statements	Reasons
1. Through S, draw a line parallel to $\overline{QT}$ that intersects $\overline{QR}$ at P: $\overline{SP} \parallel \overline{QT}$	**1.** Through a given point, only one line can be drawn parallel to a given line.
2. $\overline{QR} \parallel \overline{ST}$	**2.** Given.
3. $QPST$ is a parallelogram.	**3.** A parallelogram is a quadrilateral with two pairs of parallel sides.
4. $\overline{QT} \cong \overline{SP}$	**4.** Opposite sides of a parallelogram are congruent.
5. $\overline{QT} \cong \overline{SR}$	**5.** Given.
6. $\overline{SP} \cong \overline{SR}$	**6.** Transitive property of congruence.

(*Continued*)



Statements	Reasons
7. $\angle SPR \cong \angle R$	**7.** If two sides of a triangle are congruent, the angles opposite these sides are congruent.
8. $\angle Q \cong \angle SPR$	**8.** If two parallel lines are cut by a transversal, the corresponding angles are congruent.
9. $\angle Q \cong \angle R$	**9.** Transitive property of congruence.

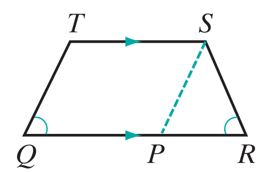
We have proved Theorem 10.20a for $\angle Q \cong \angle R$ but $\angle S$ and $\angle T$ are also congruent base angles. We often refer to $\angle Q$ and $\angle R$ as the **lower base angles** and $\angle S$ and $\angle T$ as the **upper base angles**. The proof of this theorem for $\angle S$ and $\angle T$ is left to the student. (See exercise 15.)

Theorem 10.20b If the base angles of a trapezoid are congruent, then the trapezoid is isosceles.

Given Trapezoid $QRST$ with $\overline{QR} \parallel \overline{ST}$ and $\angle Q \cong \angle R$

Prove $\overline{QT} \cong \overline{RS}$

Strategy Draw $\overline{SP} \parallel \overline{TQ}$. Prove $\angle SPR \cong \angle R$. Then use the converse of the isosceles triangle theorem.



The proof of this theorem is left to the student. (See exercise 16.) Theorems 10.20a and 10.20b can be written as a biconditional.

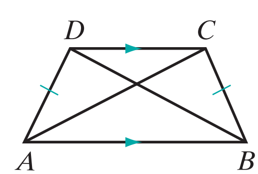
Theorem 10.20 A trapezoid is isosceles if and only if the base angles are congruent.

We can also prove theorems about the diagonals of an isosceles trapezoid.

Theorem 10.21a If a trapezoid is isosceles, then the diagonals are congruent.

Given Isosceles trapezoid $ABCD$ with $\overline{AB} \parallel \overline{CD}$ and $\overline{AD} \cong \overline{BC}$

Prove $\overline{AC} \cong \overline{BD}$

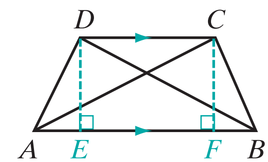


Proof We will show $\triangle DAB \cong CBA$. It is given that in trapezoid $ABCD$, $\overline{AD} \cong \overline{BC}$. It is given that $ABCD$ is an isosceles trapezoid. In an isosceles trapezoid, base angles are congruent, so $\angle DAB \cong \angle CBA$. By the reflexive property, $\overline{AB} \cong \overline{AB}$. Therefore, $\triangle DAB \cong \triangle CBA$ by SAS. Corresponding parts of congruent triangles are congruent, so $\overline{AC} \cong \overline{BD}$. ■

Theorem 10.21b

> If the diagonals of a trapezoid are congruent, then the trapezoid is isosceles.

Given Trapezoid $ABCD$ with $\overline{AB} \parallel \overline{CD}$ and $\overline{AC} \cong \overline{BD}$



Prove $\overline{AD} \cong \overline{BC}$

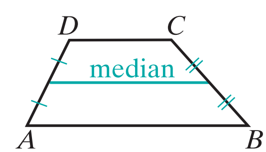
Strategy Draw $\overline{DE} \perp \overline{AB}$ and $\overline{CF} \perp \overline{AB}$. First prove that $\triangle DEB$ and $\triangle CFA$ are congruent by HL. Therefore, $\angle CAB \cong \angle DBA$. Now, prove that $\triangle ACB \cong \triangle BDA$ by SAS. Then $\overline{AD}$ and $\overline{BC}$ are congruent corresponding parts of congruent triangles.

The proof of this theorem is left to the student. (See exercise 17.) Theorems 10.21a and 10.21b can also be written as a biconditional.

Theorem 10.21

> A trapezoid is isosceles if and only if the diagonals are congruent.

Recall that the median of a triangle is a line segment from a vertex to the midpoint of the opposite sides. A triangle has three medians. A trapezoid has only one median, and it joins two midpoints.



DEFINITION

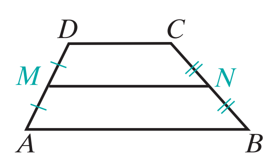
The **median of a trapezoid** is a line segment whose endpoints are the midpoints of the nonparallel sides of the trapezoid.

We can prove two theorems about the median of a trapezoid.

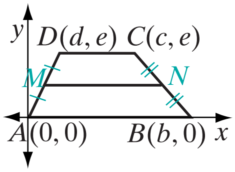
Theorem 10.22

> The median of a trapezoid is parallel to the bases.

Given Trapezoid $ABCD$ with $\overline{AB} \parallel \overline{CD}$, M the midpoint of $\overline{AD}$, and N the midpoint of $\overline{BC}$



Prove $\overline{MN} \parallel \overline{AB}$ and $\overline{MN} \parallel \overline{CD}$



Proof We will use a coordinate proof.

Place the trapezoid so that the parallel sides are on horizontal lines. Place $\overline{AB}$ on the x-axis with (0, 0) the coordinates of A and $(b, 0)$ the coordinates of B, and $b \neq 0$.

Place $\overline{CD}$ on a line parallel to the x-axis. Every point on a line parallel to the x-axis has the same y-coordinate. Let (c, e) be the coordinates of C and (d, e) be the coordinates of D, and $c, d, e \neq 0$.

Since M is the midpoint of $\overline{AD}$, the coordinates of M are

$$\left(\frac{0 + d}{2}, \frac{0 + e}{2}\right) = \left(\frac{d}{2}, \frac{e}{2}\right).$$

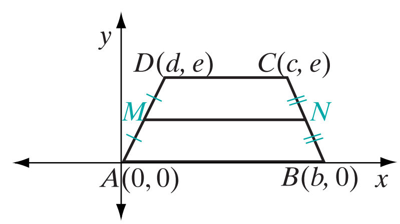
Since N is the midpoint of $\overline{BC}$, the coordinates of N are

$$\left(\frac{b + c}{2}, \frac{0 + e}{2}\right) = \left(\frac{b + c}{2}, \frac{e}{2}\right).$$

Points that have the same y-coordinate are on the same horizontal line. Therefore, $\overline{MN}$ is a horizontal line. All horizontal lines are parallel. Therefore, $\overline{MN} \parallel \overline{AB}$ and $\overline{MN} \parallel \overline{CD}$. ∎

Theorem 10.23

> The length of the median of a trapezoid is equal to one-half the sum of the lengths of the bases.



Given Trapezoid $ABCD$ with $\overline{AB} \parallel \overline{CD}$, M the midpoint of $\overline{AD}$, and N the midpoint of $\overline{CD}$

Prove $MN = \frac{1}{2}(AB + CD)$

Strategy Use a coordinate proof. Let the coordinates of A, B, C, D, M, and N be those used in the proof of Theorem 10.22. The length of a horizontal line segment is the absolute value of the difference of the x-coordinates of the endpoints.

The proof of this theorem is left to the student. (See exercise 19.)

EXAMPLE 1

The coordinates of the vertices of $ABCD$ are $A(0, 0)$, $B(4, -1)$, $C(5, 2)$, and $D(2, 6)$. Prove that $ABCD$ is a trapezoid.

Proof

$$\text{Slope of } \overline{AB} = \frac{-1 - 0}{4 - 0} = \frac{-1}{4} = -\frac{1}{4}$$

$$\text{Slope of } \overline{BC} = \frac{2 - (-1)}{5 - 4} = \frac{3}{1} = 3$$

$$\text{Slope of } \overline{CD} = \frac{6 - 2}{2 - 5} = \frac{4}{-3} = -\frac{4}{3}$$

$$\text{Slope of } \overline{DA} = \frac{0 - 6}{0 - 2} = \frac{-6}{-2} = 3$$

The slopes of $\overline{BC}$ and $\overline{DA}$ are equal. Therefore, $\overline{BC}$ and $\overline{DA}$ are parallel. The slopes of $\overline{AB}$ and $\overline{CD}$ are not equal. Therefore, $\overline{AB}$ and $\overline{CD}$ are not parallel. Because quadrilateral $ABCD$ has only one pair of parallel sides, it is a trapezoid.

EXAMPLE 2

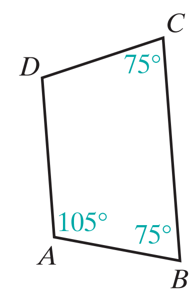
In quadrilateral $ABCD$, $m\angle A = 105$, $m\angle B = 75$, and $m\angle C = 75$.

a. Is $ABCD$ a parallelogram? Justify your answer.

b. Is $ABCD$ a trapezoid? Justify your answer.

c. Is $ABCD$ an isosceles trapezoid? Justify your answer.

d. If $AB = x$, $BC = 2x - 1$, $CD = 3x - 8$, and $DA = x + 1$, find the measure of each side of the quadrilateral.



Solution **a.** One pair of opposite angles of $ABCD$ are $\angle A$ and $\angle C$. Since these angles are not congruent, the quadrilateral is not a parallelogram. The quadrilateral does not have two pairs of parallel sides.

b. $\angle A$ and $\angle B$ are interior angles on the same side of transversal $\overline{AB}$ and they are supplementary. Therefore, $\overline{AD}$ and $\overline{BC}$ are parallel. The quadrilateral has only one pair of parallel sides and is therefore a trapezoid.

c. Because $\angle B$ and $\angle C$ are congruent base angles of the trapezoid, the trapezoid is isosceles.

d. The congruent legs of the trapezoid are $\overline{AB}$ and $\overline{CD}$.

$$\begin{aligned} AB &= CD \\ x &= 3x - 8 \\ -2x &= -8 \\ x &= 4 \end{aligned}$$

$$\begin{aligned} AB &= x \\ &= 4 \end{aligned} \qquad \begin{aligned} BC &= 2x - 1 \\ &= 2(4) - 1 \\ &= 8 - 1 \\ &= 7 \end{aligned} \qquad \begin{aligned} CD &= 3x - 8 \\ &= 3(4) - 8 \\ &= 12 - 8 \\ &= 4 \end{aligned} \qquad \begin{aligned} DA &= x + 1 \\ &= 4 + 1 \\ &= 5 \end{aligned}$$

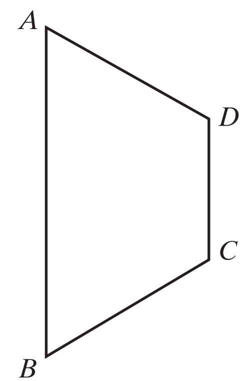
Exercises

Writing About Mathematics

1. Can a trapezoid have exactly one right angle? Justify your answer.
2. Can a trapezoid have three obtuse angles? Justify your answer.

Developing Skills

In 3–8, $ABCD$ is an isosceles trapezoid, $\overline{AB} \parallel \overline{DC}$, and $\overline{AD} \cong \overline{BC}$.



3. If $m\angle ADC = 110$, find: **a.** $m\angle BCD$ **b.** $m\angle ABC$ **c.** $m\angle DAB$.
4. If $AD = 3x + 7$ and $BC = 25$, find the value of x.
5. If $AD = 2y - 5$ and $BC = y + 3$, find AD.
6. If $m\angle DAB = 4x - 5$ and $m\angle ABC = 3x + 15$, find the measure of each angle of the trapezoid.
7. If $m\angle ADC = 4x + 20$ and $m\angle DAB = 8x - 20$, find the measure of each angle of the trapezoid.
8. The perimeter of $ABCD$ is 55 centimeters. If $AD = DC = BC$ and $AB = 2AD$, find the measure of each side of the trapezoid.

In 9–14, determine whether each statement is true or false. Justify your answer with an appropriate definition or theorem, or draw a counterexample.

9. In an isosceles trapezoid, nonparallel sides are congruent.
10. In a trapezoid, at most two sides can be congruent.
11. In a trapezoid, the base angles are always congruent.
12. The diagonals of a trapezoid are congruent if and only if the nonparallel sides of the trapezoid are congruent.
13. The sum of the measures of the angles of a trapezoid is 360°.
14. In a trapezoid, there are always two pairs of supplementary angles.

Applying Skills

15. In Theorem 10.20a, we proved that the lower base angles of $QRST$, $\angle Q$ and $\angle R$, are congruent. Use this fact to prove that the upper base angles of $QRST$, $\angle S$ and $\angle T$, are congruent.
16. Prove Theorem 10.20b, "If the base angles of a trapezoid are congruent, then the trapezoid is isosceles."
17. **a.** Prove Theorem 10.21b, "If the diagonals of a trapezoid are congruent, then the trapezoid is isosceles."

    b. Why can't Theorem 10.21b be proved using the same method as in 10.21a?

18. Prove Theorem 10.23, "The length of the median of a trapezoid is equal to one-half the sum of the lengths of the bases."

19. Let the coordinates of the vertices of $ABCD$ be $A(2, -6)$, $B(6, 2)$, $C(0, 8)$, and $D(-2, 4)$.

a. Find the slopes of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$.

b. Prove that $ABCD$ is a trapezoid.

c. Find the coordinates of E and F, the midpoints of the nonparallel sides.

d. Find the slope of $\overline{EF}$.

e. Show that the median is parallel to the bases.

20. Prove that the diagonals of a trapezoid do not bisect each other.

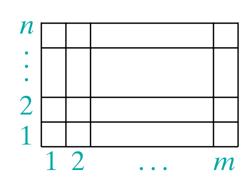
21. Prove that if the diagonals of a trapezoid are unequal, then the trapezoid is not isosceles.

22. Prove that if a quadrilateral does not have a pair of consecutive angles that are supplementary, the quadrilateral is not a trapezoid.

10-8 AREAS OF POLYGONS

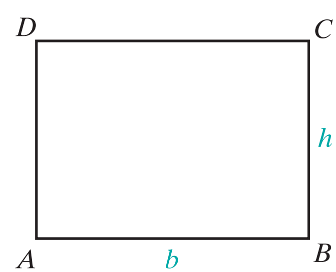
DEFINITION

The **area of a polygon** is the unique real number assigned to any polygon that indicates the number of non-overlapping square units contained in the polygon's interior.



We know that the area of a rectangle is the product of the lengths of two adjacent sides. For example, the rectangle to the right contains mn unit squares or has an area of $m \times n$ square units.

In rectangle $ABCD$, $AB = b$, the length of the base, and $BC = h$, the length of the altitude, a line segment perpendicular to the base.



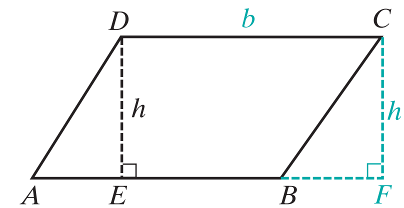
$$\text{Area of } ABCD = (AB)(BC) = bh$$

The formula for the area of every other polygon can be derived from this formula. In order to derive the formulas for the areas of other polygons from the formula for the area of a rectangle, we will use the following postulate.

Postulate 10.1 The areas of congruent figures are equal.

EXAMPLE 1

$ABCD$ is a parallelogram and E is a point on $\overline{AB}$ such that $\overline{DE} \perp \overline{AB}$. Prove that if $DC = b$ and $DE = h$, the area of parallelogram $ABCD = bh$.



Proof Let F be a point on $\overleftrightarrow{AB}$ such that $\overline{CF} \perp \overleftrightarrow{AB}$. Since two lines perpendicular to the same line are parallel, $\overline{DE} \parallel \overline{CF}$. Therefore, $EFCD$ is a parallelogram with a right angle, that is, a rectangle.

Perpendicular lines intersect to form right angles. Therefore, $\angle DEA$ and $\angle CFB$ are right angles and $\triangle DEA$ and $\triangle CFB$ are right triangles. In these right triangles, $\overline{AD} \cong \overline{BC}$ and $\overline{DE} \cong \overline{CF}$ because the opposite sides of a parallelogram are congruent. Therefore, $\triangle DEA \cong \triangle CFB$ by HL.

The base of rectangle $EFCD$ is DC. Since $ABCD$ is a parallelogram, $DC = AB = b$. The height is $DE = h$. Therefore:

$$\text{Area of rectangle } EFCD = (DC)(DE) = bh$$

$$\text{Area of parallelogram } ABCD = \text{area of } \triangle AED + \text{area of trapezoid } EBCD$$

$$\text{Area of rectangle } EFCD = \text{area of } \triangle BFC + \text{area of trapezoid } EBCD$$

$$\text{Area of } \triangle AED = \text{area of } \triangle BFC$$

Therefore, the area of parallelogram $ABCD$ is equal to the area of rectangle $EFCD$ or bh.

Exercises

Writing About Mathematics

1. If $ABCD$ and $PQRS$ are rectangles with $AB = PQ$ and $BC = QR$, do $ABCD$ and $PQRS$ have equal areas? Justify your answer.

2. If $ABCD$ and $PQRS$ are parallelograms with $AB = PQ$ and $BC = QR$, do $ABCD$ and $PQRS$ have equal areas? Justify your answer.

Developing Skills

3. Find the area of a rectangle whose vertices are (0, 0), (8, 0), (0, 5), and (8, 5).

4. **a.** Draw $\triangle ABC$. Through C, draw a line parallel to $\overleftrightarrow{AB}$, and through B, draw a line parallel to $\overleftrightarrow{AC}$. Let the point of intersection of these lines be D.

b. Prove that $\triangle ABC \cong \triangle DBC$.

c. Let E be a point on $\overleftrightarrow{AB}$ such that $\overline{CE} \perp \overleftrightarrow{AB}$. Let $AB = b$ and $CE = h$. Use the results of Example 1 to prove that the area of $\triangle ABC = \frac{1}{2}bh$.

5. Find the area of a triangle whose vertices are $(-1, -1)$, $(7, -1)$, and $(3, 5)$.

6. a. Draw trapezoid $ABCD$. Let E and F be points on $\overleftrightarrow{AB}$ such that $\overline{CE} \perp \overleftrightarrow{AB}$ and $\overline{DF} \perp \overleftrightarrow{AB}$.

b. Prove that $CE = DF$.

c. Let $AB = b_1$, $CD = b_2$, and $CE = DF = h$. Prove that the area of trapezoid $ABCD$ is $\frac{h}{2}(b_1 + b_2)$.

7. Find the area of a trapezoid whose vertices are $(-2, 1)$, $(2, 2)$, $(2, 7)$, and $(-2, 4)$.

8. a. Draw rhombus $ABCD$. Let the diagonals of $ABCD$ intersect at E.

b. Prove that $\triangle ABE \cong \triangle CBE \cong \triangle CDE \cong \triangle ADE$.

c. Let $AC = d_1$ and $BD = d_2$. Prove that the area of $\triangle ABE = \frac{1}{8}d_1d_2$.

d. Prove that the area of rhombus $ABCD = \frac{1}{2}d_1d_2$.

9. Find the area of a rhombus whose vertices are $(0, 2)$, $(2, -1)$, $(4, 2)$, and $(2, 5)$.

Applying Skills

10. a. The vertices of $ABCD$ are $A(-2, 1)$, $B(2, -2)$, $C(6, 1)$, and $D(2, 4)$. Prove that $ABCD$ is a rhombus.

b. Find the area of $ABCD$.

11. a. The vertices of $ABCD$ are $A(-2, 2)$, $B(1, -1)$, $C(4, 2)$, and $D(1, 5)$. Prove that $ABCD$ is a square.

b. Find the area of $ABCD$.

c. Find the coordinates of the vertices of $A'B'C'D'$, the image of $ABCD$ under a reflection in the y-axis.

d. What is the area of $A'B'C'D'$?

e. Let E and F be the coordinates of the fixed points under the reflection in the y-axis. Prove that $AEA'F$ is a square.

f. What is the area of $AEA'F$?

12. $\overline{KM}$ is a diagonal of parallelogram $KLMN$. The area of $\triangle KLM$ is 94.5 square inches.

a. What is the area of parallelogram $KLMN$?

b. If $MN = 21.0$ inches, what is the length of $\overline{NP}$, the perpendicular line segment from N to $\overline{KL}$?

13. $ABCD$ is a parallelogram and S and T are two points on $\overleftrightarrow{CD}$. Prove that the area of $\triangle ABS$ is equal to the area of $\triangle ABT$.

14. The vertices of $ABCD$ are $A(1, -2)$, $B(4, 2)$, $C(4, 6)$, and $D(-4, 2)$. Draw the polygon on graph paper and draw the diagonal, $\overline{DB}$.

a. Find the area of $\triangle DBC$.

b. Find the area of $\triangle DBA$.

c. Find the area of polygon $ABCD$.

15. The altitude to a base $\overline{AB}$ of trapezoid $ABCD$ is $\overline{DH}$ and the median is $\overline{EF}$. Prove that the area of a trapezoid is equal to the product, $(DH)(EF)$.

16. The vertices of polygon $PQRS$ are $P(1, 2)$, $Q(9, 1)$, $R(8, 4)$, and $S(3, 4)$. Draw a vertical line through P that intersects a horizontal line through S at M and a horizontal line through Q at N. Draw a vertical line through Q that intersects a horizontal line through R at L.

a. Find the area of rectangle $NQLM$.

b. Find the areas of $\triangle PNQ$, $\triangle QLR$, and $\triangle SMP$.

c. Find the area of polygon $PQRS$.

17. Find the area of polygon $ABCD$ if the coordinates of the vertices are $A(5, 0)$, $B(8, 2)$, $C(8, 8)$, and $D(0, 4)$.

Hands-On Activity

1. Draw square $ABCD$ with diagonals that intersect at E. Let $AC = d$. Represent BD, AE, EC, BE, and ED in terms of d.

2. Express the area of $\triangle ACD$ and of $\triangle ACB$ in terms of d.

3. Find the area of $ABCD$ in terms of d.

4. Let $AB = s$. Express the area of $ABCD$ in terms of s.

5. Write an equation that expresses the relationship between d and s.

6. Solve the equation that you wrote in step 5 for d in terms of s.

7. Use the result of step 6 to express the length of the hypotenuse of an isosceles right triangle in terms of the length of a leg.

CHAPTER SUMMARY

Definitions to Know

- A **parallelogram** is a quadrilateral in which two pairs of opposite sides are parallel.
- The **distance between two parallel lines** is the length of the perpendicular from any point on one line to the other line.
- A **rectangle** is a parallelogram containing one right angle.
- A **rhombus** is a parallelogram that has two congruent consecutive sides.
- A **square** is a rectangle that has two congruent consecutive sides.
- A **trapezoid** is a quadrilateral in which two and only two sides are parallel.
- An **isosceles trapezoid** is a trapezoid in which the nonparallel sides are congruent.
- The **area of a polygon** is the unique real number assigned to any polygon that indicates the number of non-overlapping square units contained in the polygon's interior.

Postulate

10.1 The areas of congruent figures are equal.

Theorems and Corollaries

10.1 A diagonal divides a parallelogram into two congruent triangles.
10.1a Opposite sides of a parallelogram are congruent.
10.1b Opposite angles of a parallelogram are congruent.
10.2 Two consecutive angles of a parallelogram are supplementary.
10.3 The diagonals of a parallelogram bisect each other.
10.4 If both pairs of opposite sides of a quadrilateral are congruent, the quadrilateral is a parallelogram.
10.5 If one pair of opposite sides of a quadrilateral is both congruent and parallel, the quadrilateral is a parallelogram.
10.6 If both pairs of opposite angles of a quadrilateral are congruent, the quadrilateral is a parallelogram.
10.7 If the diagonals of a quadrilateral bisect each other, the quadrilateral is a parallelogram.
10.8 All angles of a rectangle are right angles.
10.9 The diagonals of a rectangle are congruent.
10.10 If a quadrilateral is equiangular, then it is a rectangle.
10.11 If the diagonals of a parallelogram are congruent, the parallelogram is a rectangle.
10.12 All sides of a rhombus are congruent.
10.13 The diagonals of a rhombus are perpendicular to each other.
10.14 The diagonals of a rhombus bisect its angles.
10.15 If a quadrilateral is equilateral, then it is a rhombus.
10.16 If the diagonals of a parallelogram are perpendicular to each other, the parallelogram is a rhombus.
10.17 A square is an equilateral quadrilateral.
10.18 A square is a rhombus.
10.19 If one of the angles of a rhombus is a right angle, then the rhombus is a square.
10.20 A trapezoid is isosceles if and only if the base angles are congruent.
10.21 A trapezoid is isosceles if and only if the diagonals are congruent.
10.22 The median of a trapezoid is parallel to the bases.
10.23 The length of the median of a trapezoid is equal to one-half the sum of the lengths of the bases.

VOCABULARY

10-1 Quadrilateral • Consecutive vertices of a quadrilateral • Adjacent vertices of a quadrilateral • Consecutive sides of a quadrilateral • Adjacent sides of a quadrilateral • Opposite sides of a quadrilateral • Consecutive angles of a quadrilateral • Opposite angles of a quadrilateral • Diagonal of a quadrilateral

10-2 Parallelogram • Distance between two parallel lines

10-4 Rectangle

10-5 Rhombus

10-6 Square

10-7 Trapezoid • Bases of a trapezoid • Legs of a trapezoid • Isosceles trapezoid • Base angles of a trapezoid • Lower base angles • Upper base angles • Median of a trapezoid

10-8 Area of a polygon

REVIEW EXERCISES

1. Is it possible to draw a parallelogram that has only one right angle? Explain why or why not.

2. The measure of two consecutive angles of a parallelogram are represented by $3x$ and $5x - 12$. Find the measure of each angle of the parallelogram.

3. Point P is on side $\overline{BC}$ of rectangle $ABCD$. If $AB = BP$, find m$\angle APC$.

4. Quadrilateral $ABCD$ is a parallelogram, M is a point on $\overline{AB}$, and N is a point on $\overline{CD}$. If $\overline{DM} \perp \overline{AB}$ and $\overline{BN} \perp \overline{CD}$, prove that $\triangle AMD \cong \triangle CNB$.

5. Quadrilateral $ABCD$ is a parallelogram, M is a point on $\overline{AB}$, and N is a point on $\overline{CD}$. If $\overline{CM} \parallel \overline{AN}$, prove that $\triangle AND \cong \triangle CMB$.

6. The diagonals of rhombus $ABCD$ intersect at E.

a. Name three angles that are congruent to $\angle EAB$.

b. Name four angles that are the complements of $\angle EAB$.

7. The diagonals of parallelogram $PQRS$ intersect at T. If $\triangle PTQ$ is isosceles with $\angle PTQ$ the vertex angle, prove that $PQRS$ is a rectangle.

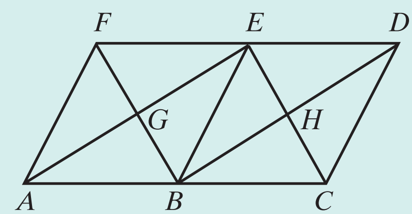
8. Point P is the midpoint of side $\overline{BC}$ of rectangle $ABCD$. Prove that $\overline{AP} \cong \overline{DP}$.

9. A regular polygon is a polygon that is both equilateral and equiangular.

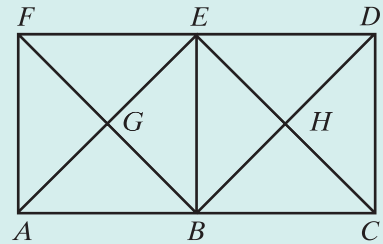
a. Is an equilateral triangle a regular polygon? Justify your answer.

b. Is an equilateral quadrilateral a regular polygon? Justify your answer.

10. The diagonals of rhombus $ABEF$ intersect at G and the diagonals of rhombus $BCDE$ intersect at H. Prove that $BHEG$ is a rectangle.



11. The diagonals of square $ABEF$ intersect at G and the diagonals of square $BCDE$ intersect at H. Prove that $BHEG$ is a square.



12. Points $A(-3, -2)$, $B(3, -2)$, $C(5, 3)$, and $D(-1, 3)$ are the vertices of quadrilateral $ABCD$.

a. Plot these points on graph paper and draw the quadrilateral.

b. What kind of quadrilateral is $ABCD$? Justify your answer.

c. Find the area of quadrilateral $ABCD$.

13. The vertices of quadrilateral $DEFG$ are $D(1, -1)$, $E(4, 1)$, $F(1, 3)$, and $G(-2, 1)$.

a. Is the quadrilateral a parallelogram? Justify your answer.

b. Is the quadrilateral a rhombus? Justify your answer.

c. Is the quadrilateral a square? Justify your answer.

d. Explain how the diagonals can be used to find the area of the quadrilateral.

14. The coordinates of the vertices of quadrilateral $ABCD$ are $A(0, 0)$, $B(2b, 0)$, $C(2b + 2d, 2a)$, and $D(2d, 2a)$.

a. Prove that $ABCD$ is a parallelogram.

b. The midpoints of the sides of $ABCD$ are P, Q, R, and S. Find the coordinates of these midpoints.

c. Prove that $PQRS$ is a parallelogram.

15. The area of a rectangle is 12 square centimeters and the perimeter is 16 centimeters.

a. Write an equation for the area of the rectangle in terms of the length, x, and the width, y.

b. Write an equation for the perimeter of the rectangle in terms of the length, x, and the width, y.

c. Solve the equations that you wrote in **a** and **b** to find the length and the width of the rectangle.

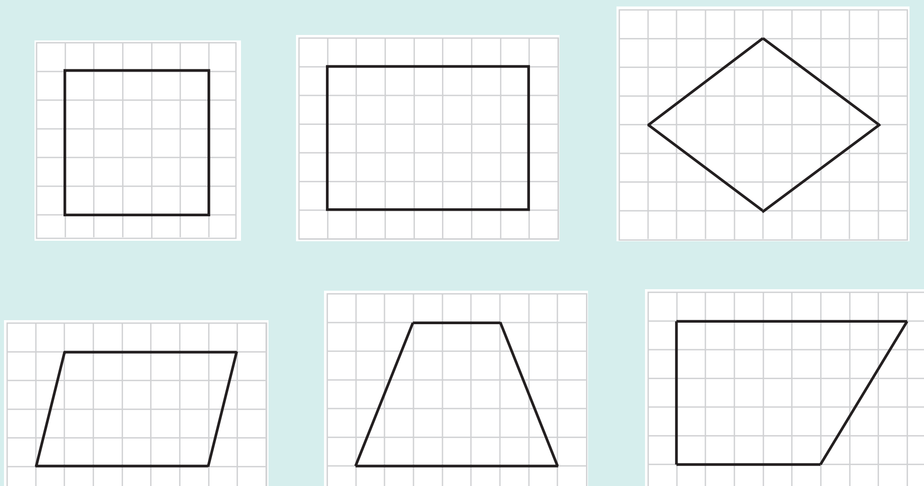
16. Each of the four sides of quadrilateral $ABCD$ is congruent to the corresponding side of quadrilateral $PQRS$ and $\angle A \cong \angle P$. Prove that $ABCD$ and $PQRS$ are congruent quadrilaterals or draw a counterexample to prove that they may not be congruent.

Exploration

In Chapter 5 we found that the perpendicular bisectors of the sides of a triangle intersect in a point called the circumcenter. Do quadrilaterals also have circumcenters?

In this activity, you will explore the perpendicular bisectors of the sides of quadrilaterals. You may use compass and straightedge or geometry software.

a. Construct the perpendicular bisectors of two adjacent sides of each of the following quadrilaterals.



b. Construct the third and fourth perpendicular bisectors of the sides of each of the quadrilaterals. For which of the above quadrilaterals is the intersection of the first two perpendicular bisectors the same point as the intersection of the last two perpendicular bisectors?

c. When all four perpendicular bisectors of a quadrilateral intersect in the same point, that point is called the circumcenter. Of the specific quadrilaterals studied in this chapter, which types do you expect to have a circumcenter and which types do you expect not to have a circumcenter?

d. For each of the quadrilaterals above that has a circumcenter, place the point of your compass on the circumcenter and the pencil on any of the vertices. Draw a circle.

e. Each circle drawn in **d** is called the **circumcircle** of the polygon. Based on your observations, write a definition of circumcircle.

CUMULATIVE REVIEW CHAPTERS 1–10

Part I

Answer all questions in this part. Each correct answer will receive 2 credits. No partial credit will be allowed.

1. The measure of an exterior angle at vertex B of isosceles $\triangle ABC$ is 70°. The measure of a base angle of the triangle is

(1) 110° (2) 70° (3) 55° (4) 35°

2. The measure of $\angle A$ is 12 degrees less than twice the measure of its complement. The measure of the $\angle A$ is

(1) 34° (2) 56° (3) 64° (4) 116°

3. The slope of the line determined by $A(2, -3)$ and $B(-1, 3)$ is

(1) –2 (2) $-\frac{1}{2}$ (3) $\frac{1}{2}$ (4) 2

4. What is the slope of a line that is parallel to the line whose equation is $3x + y = 5$?

(1) –3 (2) $-\frac{5}{3}$ (3) $\frac{5}{3}$ (4) 3

5. The coordinates of the image of $A(3, -2)$ under a reflection in the x-axis are

(1) (3, 2) (2) (−3, 2) (3) (−3, −2) (4) (−2, 3)

6. The measures of two sides of a triangle are 8 and 12. The measure of the third side *cannot* be

(1) 16 (2) 12 (3) 8 (4) 4

7. The line segment $\overline{BD}$ is the median and the altitude of $\triangle ABC$. Which of the following statements must be false?

(1) $\overline{BD}$ bisects $\overline{AC}$.
(2) $\triangle BDA$ is a right triangle.
(3) $m\angle A = 90$
(4) B is equidistant from A and C.

8. What is the equation of the line through $(0, -1)$ and perpendicular to the line whose equation is $y = 2x + 5$?

(1) $y = 2x - 1$
(2) $x + 2y + 2 = 0$
(3) $2y - 1 = x$
(4) $2x + y + 2 = 0$

9. Which of the following transformations is not a direct isometry?

(1) line reflection
(2) point reflection
(3) translation
(4) rotation

10. If $\overline{AC} \cong \overline{DF}$ and $\angle A \cong \angle D$, which of the following is not sufficient to prove that $\triangle ABC \cong \triangle DEF$?

(1) $\overline{AB} \cong \overline{DE}$ (2) $\overline{BC} \cong \overline{EF}$ (3) $\angle C \cong \angle F$ (4) $\angle B \cong \angle E$

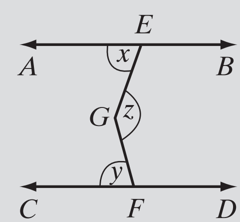
Part II

Answer all questions in this part. Each correct answer will receive 2 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

11. The measures of the angles of a triangle can be represented by $3x$, $4x + 5$, and $5x - 17$. Find the measure of each angle of the triangle.

12. In the diagram, $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$. Prove that

$m\angle x + m\angle y = m\angle z$.



Part III

Answer all questions in this part. Each correct answer will receive 4 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

13. The bases, $\overline{AB}$ and $\overline{DE}$, of two isosceles triangles, $\triangle ABC$ and $\triangle DEF$, are congruent. If $\angle CAB \cong \angle FDE$, prove that $\triangle ABC \cong \triangle DEF$.

14. Points A, B, C, and D are on a circle with center at O and diameter $\overline{AOB}$. $ABCD$ is a trapezoid with $\overline{AB} \parallel \overline{CD}$ and $\overline{AO} \cong \overline{CD}$. Prove that $\overline{OC}$ and $\overline{OD}$ separate the trapezoid into three equilateral triangles. (*Hint:* All radii of a circle are congruent.)

Part IV

Answer all questions in this part. Each correct answer will receive 6 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

15. The vertices of quadrilateral $KLMN$ are $K(-3, 1)$, $L(2, 0)$, $M(6, 4)$, and $N(2, 6)$. Show that $KLMN$ is a trapezoid.

16. The coordinates of the vertices of $\triangle ABC$ are $A(2, 2)$, $B(4, 0)$, and $C(4, 2)$. Find the coordinates of the vertices of $\triangle A'B'C'$, the image of $\triangle ABC$ under the composition $r_{y\text{-axis}} \circ r_{y=x}$.